

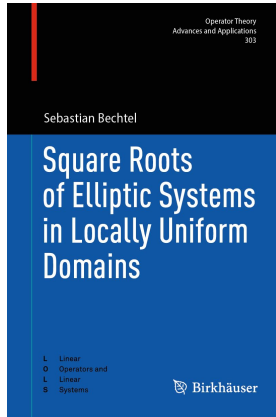
# What we know about square roots of elliptic systems... and a bit more!

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joint work (in part) with  
R. Brown (Lexington), M. Egert, R. Haller (Darmstadt), P. Tolksdorf (KIT)



# Elliptic systems

Fix  $d$  (dimension) and  $m$  (system size). Let  $O \subseteq \mathbb{R}^d$ . Consider (formally):

$$L_0 = -\operatorname{div}(A\nabla + b) + c \cdot \nabla + d$$



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$$a : V \times V \rightarrow \mathbb{C}, \quad a(u, v) = \int_O \begin{bmatrix} d & c \\ b & A \end{bmatrix} \begin{bmatrix} u \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \nabla v \end{bmatrix}} dx$$

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Examples for  $V$ :  $H_0^1(O)$  (Dirichlet BC),  $H^1(O)$  (natural BC),  
 $H_D^1(O)$  (mixed BC).



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Recall: sesquilinear form

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with complex, bounded, measurable & elliptic coefficients.

**Associate** operator  $L_0$  in  $L^2(O)$  with  $\alpha$  via

$$u \in D(L_0) \text{ \& } L_0 u = w \in L^2(O) \quad \Leftrightarrow \quad (w | v) = a(u, v) \quad \text{for all } v \in V.$$



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$L$  is **sectorial** of angle  $\omega \in [0, \pi)$ : still nice, but maybe no generator of semigroup.



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Geometry & boundary conditions: counterexamples/restrictions by Jerison–Kenig, Shamir, ...



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- $u' = A^{-1} = 1 - \frac{1}{2}1_{[0,1]}$  on  $[0, 2] \implies u' \notin H^1(\mathbb{R})$ , **no optimal elliptic regularity!**



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## 1 Quasilinear problems

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## 2 Elliptic BVP over Lipschitz graph



# What is Kato's square root problem?

$L$  sectorial operator  $\implies$  exists operator  $L^{\frac{1}{2}}$  such that  $L = L^{\frac{1}{2}} L^{\frac{1}{2}}$  (by functional calculus). Call  $L^{\frac{1}{2}}$  the **square root** of  $L$ .



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## Conjecture

*One has*

$$D(L^{\frac{1}{2}}) = V \quad \& \quad \|L^{\frac{1}{2}} u\|_{L^2(O)} \approx \|u\|_V.$$

*Implicit constants depend on ellipticity, dimensions and  $V$  only.*



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True for  $V = H^1(\mathbb{R}^n)$ : Auscher–Hofmann–Lacey–McIntosh–Tchamitchian (Ann. Math. '02)



# Geometrical framework

Let  $O \subseteq \mathbb{R}^d$  open,  $D \subseteq \partial O$ ,  $N = \partial O \setminus D$ .



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# Solution to Kato's square root problem

Theorem (B., Egert, Haller-Dintelmann, Adv. Math. '20)

*Let  $O \subseteq \mathbb{R}^d$  open,  $D \subseteq \partial O$ ,  $N = \partial O \setminus D$ . Assume  $D$  is Ahlfors–David regular and  $O$  is locally uniform near  $N$ . Then one has*

$$D(L^{\frac{1}{2}}) = H_D^1(O) \quad \& \quad \|L^{\frac{1}{2}}u\|_{L^2(O)} \approx \|u\|_{H^1(O)}.$$

*Implicit constants depend on ellipticity, dimensions and geometry only.*



# What about $L^p$ -theory?

Reformulation of square root property:

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# Critical numbers

Define

$$I(L) = \{p \in (1, \infty) : \{(1 + t^2 L)^{-1}\}_{t>0} \text{ is } L^p\text{-bounded}\},$$
$$J(L) = \{p \in (1, \infty) : \mathcal{L}_0 \text{ is } p\text{-isomorphism}\}.$$

Put

$$p_-(L) = \inf I(L), \quad p_+(L) = \sup I(L),$$
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- In general:  $\tilde{q}_+(L) \leq 2 + \varepsilon$ .



## Theorem (B., JDE '24 + monograph)

*Assume setting of  $L^2$ -result.*

*Let  $p \in (p_-(L), \tilde{q}_+(L))$ . Then  $L^{\frac{1}{2}}: W_D^{1,p}(O) \rightarrow L^p(O)$  isomorphism.*

The interval  $(p_-(L), \tilde{q}_+(L))$  is (essentially) optimal.



# Consequences of square root estimates

## 1 Trace estimate for non-autonomous problems

For  $f \in L^2([0, T]; L^2(O))$  consider

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Important for applications to **quasilinear problems** with **Neumann BC!**  
(current project with M. Veraar)



# Consequences of square root estimates

## ② Maximal regularity / Bounded $H^\infty$ -calculus over $W^{-1,p}$

Application to evolution equations with **surface densities**: need forcing term in distribution space  $W^{-1,p}$ ,  $p > d$ .



# Consequences of square root estimates

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Condition  $p > d$  fulfilled for:

$A$  real,  $m = 1$   
any dimension  $d$

general complex  $A$   
 $d \leq 4$



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## ③ Lipschitz dependence of $L^{\frac{1}{2}}$ on coefficients

Let  $L_A = -\operatorname{div}(A\nabla)$ ,  $L_B = -\operatorname{div}(B\nabla)$  (say with real coefficients). Does one have

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**But:** Need complex structure (for analyticity) even for real statement!



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# Open questions / Ongoing projects

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Current project with M. Egert and B. Kosmela (Darmstadt).



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However: proof for Kato uses auxiliary first-order operators: **first-order approach**, Axelsson–Keith–McIntosh (Invent. Math. '06).



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Ongoing project: Kato for Robin BC with A. Rosén (formerly: Axelsson) using second-order approach



¡Gracias por su atención!

A digital version of this presentation can be found here:

