Square roots of elliptic systems on open sets

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Introduction

Let $O \subseteq \mathbb{R}^d$ open, $D \subseteq \partial O$ closed. Consider (formally):

$$L = -\operatorname{div} A\nabla - \operatorname{div} b + c\nabla + d \quad \text{on } O \quad \text{subject to } u = 0 \text{ in } D$$

More precisely:

- $W^{1,2}_D(O) \subseteq W^{1,2}(O)$ closed subspace with "vanishing trace" in D
- $a_{ij}, b_i, c_j, d : O \longrightarrow \mathcal{L}(\mathbb{C}^m)$, $m \geq 1$, measurable, bounded, elliptic
- define

$$a : W^{1,2}_D(O)^m \times W^{1,2}_D(O)^m \longrightarrow \mathbb{C}, \qquad a(u,v) = \int_O \begin{bmatrix} d & c \\ b & A \end{bmatrix} \begin{bmatrix} u \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \nabla v \end{bmatrix}} \, dx$$

- $\mathcal{L} : W^{1,2}_D(O)^m \xrightarrow{\cong} (W^{1,2}_D(O)^m)^*$, $\langle \mathcal{L}u, v \rangle = a(u,v)$
- L restriction of $\mathcal{L}$ to $L^2(O)^m \subseteq (W^{1,2}_D(O)^m)^*$.

<u>good properties of L:</u> densely defined, invertible, maximal accretive, sectorial, ...

<u>problem:</u> regularity of solutions?

α	$D(L^\alpha)$	
$\alpha \in (0, \frac{1}{2})$	$D(L^\alpha) = [L^2(O)^m, W_D^{1,2}(O)^m]_{2\alpha} \subseteq W^{2\alpha,2}(O)^m$	(Kato '61)
$\alpha = \frac{1}{2}$	?	$\longrightarrow$ "Kato's square root problem"
$\alpha \in (\frac{1}{2}, 1]$	$D(L^\alpha) \neq W_D^{2\alpha,2}(O)^m$.	(e.g. counterexample in $d=1$ by interpolation)

<u>Goal:</u> show $D(L^{1/2}) = W_D^{1,2}(O)^m$ (with equivalent norms)

Kato's square root problem: a (very short) history

- '61 & '62: question posed by Kato, refined by Lions
- '01: Auscher, Hofmann, Lacey, McIntosh, Tchamitchian (Annals of Math)

  whole space case
- '06: Axelsson (Rosén), Keith, McIntosh (Inventiones)

  first-order reformulation and application to special Lipschitz domains

  (for convenience: $L = -\operatorname{div} A \nabla$). Put $D = \begin{bmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{bmatrix}$, $B = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & A \end{bmatrix}$.

  $$\Rightarrow (BD)^2 = \begin{bmatrix} L & 0 \\ 0 & * \end{bmatrix}, \quad |BD| := \sqrt{(BD)^2} = \begin{bmatrix} \sqrt{L} & 0 \\ 0 & * \end{bmatrix}$$

  bdd. H^∞-calc. for $BD \Rightarrow \|\sqrt{L}u\| = \| \, |BD| \begin{bmatrix} u \\ 0 \end{bmatrix} \| \overset{H^\infty\text{-calc.}}{\approx} \| BD \begin{bmatrix} u \\ 0 \end{bmatrix} \| \approx \|\nabla u\|$
- '14: Egert, Haller-Dintelmann, Tolksdorf (JFA) $\longrightarrow$ next slide

Kato in rough domains

Theorem: (Egert, Haller-Dintelmann, Tolksdorf '14)

Assume:

- O bounded domain
- O interior thick
- D Ahlfors–David regular
- Lipschitz charts around $N = \partial O \setminus D$.

Then: $D(L^{1/2}) = W_D^{1,2}(O)^m$ with equivalent norms.

How to get rid of thickness?

Observation: interior thick $\Rightarrow$ dyadic structure

(crucial for involved harmonic analysis!)

Idea: Extend L to system on interior thick set!

Proposition:

$(\mathbf{O}, \mathbf{D}, \mathbf{L})$ has Kato property $\iff$ (O_i, D_i, L_i) has Kato property (with uniform constants)

Hence: Theorem with ITC $\implies$ $(\mathbf{O}, \mathbf{D}, \mathbf{L})$ has Kato

$\Downarrow$ Prop.

Theorem without ITC $\impliedby$ (O, D, L) has Kato

Observation: Kato $\Longrightarrow$ Riesz transform $\nabla L^{-1/2}$ bounded, where underlying metric measure space <u>not</u> doubling.

What about L^p-theory?

<u>Kato:</u> $\quad L^{1/2} : W_D^{1,2}(O)^m \xrightarrow{\cong} L^2(O)^m$

<u>Question:</u> extrapolation to $W_D^{1,p}(O)^m \xrightarrow{\cong} L^p(O)^m$ for which p?

Some critical numbers:

- $p_-(L) = \inf \mathcal{J}(L)$
- $p_+(L) = \sup \mathcal{J}(L)$
- $q_-(L) = \inf \mathcal{J}(L)$
- $q_-(L) = \sup \mathcal{J}(L)$

where:

- $\mathcal{J}(L) = \{ p \in (1,\infty) : \ e^{-tL} \ L^p\text{-bounded} \}$
- $\mathcal{J}(L) = \{ q \in (1,\infty) : \ t\nabla e^{-t^2 L} \ L^q\text{-bounded} \}$

Known results:

- on $\mathbb{R}^d$: Auscher : $p \in (p_-(L), q_+(L))$
- bdd. domain, real & scalar coefficients:

  Auscher, Badr, Haller-Dintelmann, Rehberg : $p \in (1, 2] = (p_-(L), 2]$
- bdd domain, complex systems:

  Egert : $p \in (p_-(L), 2+\varepsilon)$, ε depending on ellipticity.

Goals:

1. geometry as in first part of talk
2. optimal & explicit upper endpoint
3. sharpness at endpoints

Characterization of upper endpoint

problem with $q_+(L)$: related to boundedness of Riesz transform

argument of "homogeneous" nature using

conservation property

Instead, consider diagram:

$$\begin{tikzcd}
W_D^{1,2}(O)^m \arrow[d, "\mathcal{L}"] \arrow[dr, "L^{1/2}"] & \\
(W_D^{1,2}(O)^m)^* & L^2(O)^m \arrow[l, "L^{1/2}"]
\end{tikzcd}$$

Heuristic: 2 arrows p-isomorphisms

$\Longrightarrow$ 3 arrows p-isomorphisms

$\longrightarrow$ new critical number: $\tilde{q}_+(L) = \sup \{ p \in (1,\infty) : \mathcal{L}$ is (compatible) p-isomorphism $\}$

Let $p \in (2, \tilde{q}_+(L))$.

$L^{1/2}$ p-isomorphism reduces to $L^{1/2} : L^p(O)^m \xrightarrow{\cong} (W_D^{1,p}(O)^m)^*$

$\overset{\text{duality}}{\iff}$ $(L^*)^{1/2}$ p'-isomorphism

$\Longleftarrow$ $p_-(L) < p' \leq 2$

$\iff$ $2 \leq p < p_+(L)$.

Hence: need to show $p_+(L) > \tilde{q}_+(L)$.

Lemma: $p_+(L) \geq (\tilde{q}_+(L))^*$.

Sketch of proof:

$$2 \leq q \leq \tilde{q}_+(L)^* \implies \begin{array}{l} \text{a) } \mathcal{L} \text{ is } q_*\text{-isomorphism} \\ \text{b) } H^\infty\text{-calc. of } L \text{ is } q_{**}\text{-bounded} \end{array}$$

Calculate:

$$\|e^{-tL}u\|_q \overset{\text{Sobolev}}{\lesssim} \|e^{-tL}u\|_{W^{1,q_*}} = \|\mathcal{L}^{-1}Le^{-tL}u\|_{W^{1,q_*}}$$

$$\overset{q_*\text{-iso}}{\lesssim} \|Le^{-tL}u\|_{W^{-1,q_*}} \overset{\text{Sobolev}}{\lesssim} t^{-1}\|tLe^{-tL}u\|_{q_{**}}$$

$$\overset{H^\infty\text{-calc.}}{\lesssim} t^{-1}\|u\|_{q_{**}}$$

$\implies e^{-tL}$ hypercontractive $\implies e^{-tL}$ L^p-bounded for $p\in(2,q)$. $\square$

Sharpness at endpoints

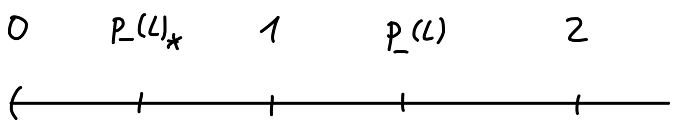
Summary:
- $p \in (p_-(L), \tilde{q}_+(L)) \implies L^{1/2}$ p-isomorphism
- $p \notin [p_-(L), \tilde{q}_+(L)] \implies L^{1/2}$ <u>not</u> p-isomorphism

Remaining question: What happens when (for example) $p = p_-(L)$?

That is to say: $\{p \in (1,\infty) : L^{1/2}$ p-isomorphism$\}$ open in $(1,\infty)$?

Argument for lower endpoint:

Assume $p_-(L) > 1$ & $L^{1/2}$ $p_-(L)$-isomorphism.



$\overset{\text{Sneiberg}}{\implies}$ $L^{1/2}$ $(p_-(L) - \varepsilon)$-isomorphism

$\lightning$

Thank you

for your attention!