

Well-posedness of quasilinear PDEs with critical data

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Our quasilinear problem

Quasilinear problem on $(0, \infty) \times \mathbb{R}^n$:

$$\partial_t u - \operatorname{div}(a(u) \nabla u) = 0, \quad u(0) = u_0. \quad (\text{QL})$$



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Examples for X : L^∞ , BMO , BUC , ...



Was there a little bias?



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Harnack principle:

$$w_0 \in BUC \Leftrightarrow w \in BUC.$$

Consequences for u : if $u_0 \in BUC$, we have

- additional regularity for solution & coefficients,
- strong continuity (fixed-point, uniqueness),
- no blow-up $\rightsquigarrow$ global solutions.



Theorem

Given $u_0 \in BUC$, we have existence, uniqueness and regularity of global, bounded weak solutions to (QL).



Linearization for fixed-point argument:

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We use (QL) to **derive** good control for ∇u !



Get control for ∇u

Take ψ with $\psi = 1$ on $(\frac{1}{2}, 1) \times B(0, 1)$ and $\text{supp}(\psi) \subseteq (\frac{1}{4}, 1) \times B(0, 2)$.



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Then: $v = w\psi$ solves

$$\partial_t v - \text{div}(A \nabla v) = v \partial_t \psi - A \nabla v \cdot \nabla \psi + \text{div}(v A \nabla \psi).$$



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L^q -theory (if $A \in BUC$):

$$\left(\int_{\frac{1}{2}}^1 \int_{B(0,1)} |\nabla w|^q \right)^{\frac{1}{q}} \lesssim \left(\int_{\frac{1}{4}}^1 \int_{B(0,2)} |\nabla w|^{q_*} \right)^{\frac{1}{q_*}} + \|w\|_{\infty}.$$



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Finally: iterate + Caccioppoli + scaling/transformation:

$$\sup_{t>0, x \in \mathbb{R}^n} \left(\int_{\frac{t}{2}}^t \int_{B(x, \sqrt{t})} |\sqrt{s} \nabla w|^q \right)^{\frac{1}{q}} \lesssim \|w\|_{\infty}.$$



Definition

Let $p, q \in (1, \infty]$ and $\beta \in \mathbb{R}$. The space $Z_\beta^{p,q}$ consists of all $f: \mathbb{R}_+^{1+n} \rightarrow \mathbb{R}$ such that

$$\|f\|_{Z_\beta^{p,q}} := \left(\iint_{\mathbb{R}_+^{1+n}} \left(\int_{t/2}^t \int_{B(x, \sqrt{t})} |s^{-\beta} f(s, y)|^q dy ds \right)^{\frac{1}{q}} dx \frac{dt}{t} \right)^{\frac{1}{p}} < \infty.$$



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In this language:

$$\|\nabla w\|_{Z_{-1/2}^{\infty,q}} \lesssim \|w\|_\infty \leq \|w_0\|_\infty, \quad q \in (1, \infty).$$



Reminder: linearization for (QL)

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Autonomous linear estimate: let $A_0 \in BUC$, for instance $A_0(x) = a(u_0(x))$.

Put solution operator $S(F) = v$, where v solution to

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Mapping properties:

- singular case: $\nabla S : Z_{-1/2}^{\infty,q} \rightarrow Z_{-1/2}^{\infty,q}$ for all $q \in (1, \infty)$,
- non-singular case: $S : Z_{-1/2}^{\infty,q} \rightarrow L^\infty$ for all $q \in (n+2, \infty)$.



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Local existence for (QL) on fixed-point space

$$X_{u_0}(\lambda, T, q) = \{v \mid \|v - u_0\|_{\infty, T} \leq \lambda, \|\nabla v\|_{Z_{-1/2}^{\infty,q}(T)} \leq \lambda\}.$$



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Strong continuity + perturbation.



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Hence: u is **global**!



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- $u \in BUC$ with $\|u\|_\infty \leq \|u_0\|_\infty$ and $\text{osc}(u)$ controlled by $\text{osc}(u_0)$,



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- Carleson condition:

$$\sup_{t>0, x \in \mathbb{R}^n} \left(\int_0^t \int_{B(x, \sqrt{t})} |\nabla u|^2 \right)^{\frac{1}{2}} \lesssim \|u_0\|_\infty.$$



Château du Haut-Buc



Thanks for your attention!

A digital version of this presentation can be found here:

