Existence for quasilinear systems of SPDEs in a variational setting

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Our problem for today

Consider SPDE

$$\begin{aligned} du &= \left[\partial_i(a_{ij}(u)\partial_j u) + \partial_i \Phi_i(u) + \phi(u)\right] dt \\ &\quad + \sum_{n\geq 1} \left[b_{n,j}(u)\partial_j u + g_n(u)\right] dw_n, \\ u(0) &= u_0, \end{aligned}$$

on $D \subseteq \mathbb{R}^d$ bounded & subject to Dirichlet BC.

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on $D \subseteq \mathbb{R}^d$ bounded & subject to Dirichlet BC, $\alpha = 1, \ldots, N$.

Coefficients $a_{ij}^{\alpha\beta}$, $b_{n,j}^{\alpha\beta}$ are symmetric, no smoothness, and elliptic:

$$\left(a_{ij}^{\alpha\beta}(t,x,y) - \frac{1}{2} b_{n,i}^{\gamma\alpha}(t,x,y^\alpha) b_{n,j}^{\gamma\beta}(t,x,y^\alpha)\right)\xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2.$$

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Question

Does a solution to this system of SPDEs exist?

Some inspiration from the deterministic world

Deterministic problem (à la Disser, ter Elst, Rehberg JDE '17)

$$u' - \partial_i(a_{ij}(u)\partial_j u) = \partial_i \Phi_i(u) + \phi(u),$$
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Tools used:

1. well-posedness for linear equation with $f \in L^p(0, T; H^{-1}(D))$ where $p > 2$,
2. Schauder's fixed point theorem.

deterministic extrapolation result . . .

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. . . and why it fails for SPDEs!

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For $p \in (1, \infty)$ put

$$E_p = L^p(0, T; H^{-1}(D)),$$
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Parabolic operator

$$\partial_t - \partial_i(a_{ij}\partial_j) \colon V_p \to E_p$$

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Moreover: $\partial_t - \partial_i(a_{ij}\partial_j)$ invertible $V_2 \to E_2$ by Lax–Milgram lemma.

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Fact: $(E_p)_{p\in(1,\infty)}$ and $(V_p)_{p\in(1,\infty)}$ are complex interpolation scales

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Lemma (Sneiberg)

Let T bounded between interpolation scales $(X_i)_{i\in(a,b)}$ and $(Y_i)_{i\in(a,b)}$. If $T\colon X_{i_*} \to Y_{i_*}$ invertible, then $T\colon X_i \to Y_i$ invertible for all $i \in (i_* - \varepsilon, i_* + \varepsilon)$.

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Upshot: $\partial_t-\partial_i(a_{ij}\partial_j)$ invertible $V_{2+\varepsilon}\to E_{2+\varepsilon}$.

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But: Set $E_p = L^p(\Omega \times (0,T); H^{-1}(D)) \times L^p(\Omega \times (0,T); \mathcal{L}_2(U, L^2(D)))$,
V_p analogous, solution operator

$$S\colon E_2 \ni (f,g) \mapsto u \in V_2$$

bounded, linear.

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Here we can attack! (later)

recap: stochastic compactness method

Blueprint of stochastic compactness method (for example Debussche–Hofmanova–Vovelle)

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4. Identify $\widetilde{u}$ as solution of original SPDE.

Approximation of Debussche–Hofmanova–Vovelle

Approximate second order SPDE by fourth order SPDEs.

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If only we could extrapolate variational regularity . . .


YES, WE CAN!

extrapolation of variational solutions

Let

- $V \subseteq H \subseteq V^*$ Gelfand triple,
- W a U-cylindrical Brownian motion,
- $A\colon \Omega \times (0, T) \to \mathcal{L}(V, V^*)$ symmetric, bounded,
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Only assume ellipticity:

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(Can reduce to $M = 0$.)

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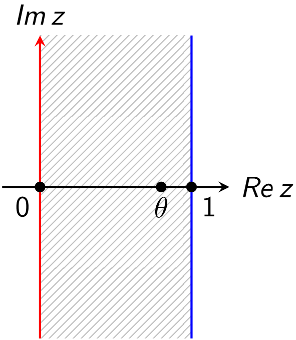
- $S \subseteq \mathbb{C}$ unit strip.
- Craft $S \ni z \mapsto (A(z), B(z))$ analytic
- with $A(\theta) = A$ and $B(\theta) = B$ for $\theta \in (0, 1)$.



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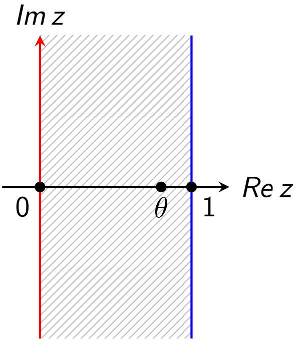

Red line: perturbation of autonomous case
$\rightsquigarrow$ L^p-maximal regularity for all p

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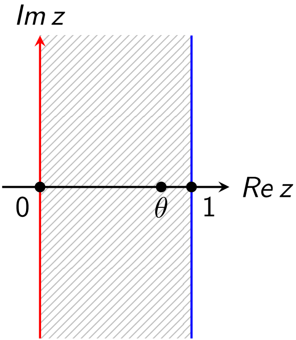

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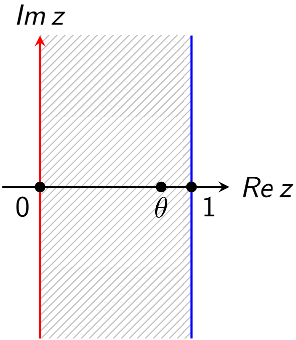

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Upshot: Stein interpolation of solution operator between L^2 and L^p
$\implies$ $L^{2+\varepsilon}$-maximal regularity for $(A(\theta), B(\theta)) = (A, B)$.

extrapolation of variational solutions – main results

Theorem (extrapolated regularity – B., Veraar)

Exists $p > 2$ depending on ellipticity of (A, B) such that for

$$f \in L^p(\Omega \times (0,T); V^*),\ g \in L^p(\Omega \times (0,T); \mathcal{L}_2(U,H)),$$
$$u_0 \in L^p(\Omega; (H,V)_{1-2/p,p})$$

unique variational solution u satisfies

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Theorem (universal compactness – B., Veraar)

Suppose $V \subseteq H$ compact. The laws of

$$\{u \text{ solution} : (A, B) \text{ uniformly elliptic}, \|f\|, \|g\|, \|u_0\| \leq K\}$$

are tight on $C([0, T]; H)$.

back to the start

Recall our system of SPDEs with Dirichlet BC

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Universal compactness result + stochastic compactness method

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a.s.\ \widetilde{u}_n \to \widetilde{u} \text{ in } C([0,T];L^2(D)) \quad \& \quad \nabla\widetilde{u}_n \rightharpoonup \nabla\widetilde{u} \text{ in } L^p(0,T;L^2(D)).
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Latter fact in general not useful, but:

$\nabla\widetilde{u}_n$ bounded in $L^p(\Omega\times(0,T);L^2(D)) \implies$ Vitali's convergence theorem applicable

existence results on quasilinear SPDEs

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Theorem (B., Veraar)

Let Φ, ϕ, g Lipschitz, $u_0 \in L^p(\Omega; B^{1-2/p}_{2,p,0}(D)) \implies$ system admits solution.

With some further (mild) assumptions: Φ, ϕ of polynomial growth possible.

Thank you for your attention!

A digital version of this presentation can be found here: