# Hardy spaces on open sets

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25th of January, 2024

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Maximal function characterization and square function characterization ($\rightarrow$ area integral).

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Atomic description:

$$f \in H^1(\mathbb{R}^d) \quad \Leftrightarrow \quad f = \sum_j \lambda_j a_j,$$

where $(\lambda_j)_j$ summable and $(a_j)_j$ are "atoms": satisfy localization, size and cancellation conditions.

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Observe: conditions independent of $-\Delta$!

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Can atomic spaces save us?

# BVPs on cylindrical domains

Let $O \subseteq \mathbb{R}^d$ open. Consider

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What about atomic description?

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Hints from the literature (say when $O$ Lipschitz domain):

| Dirichlet | Neumann |
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| $H^1_{-\Delta_0}$ | |
| $H^1_{Mi}$ | |
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| $H^1_r$ | $H^1_z$ |

- $H^1_{Mi}$ by Miyachi: additional “boundary atoms” without cancellation,
- $H^1_{CW}$ by Coifman–Weiss: classical atoms on $O$.

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- allow $O$ non-connected,
- mixed boundary conditions: Dirichlet BC only on $D \subseteq \partial O$.

Also: mixed BC approach unifies cases of Dirichlet and Neumann BC :)

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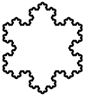
To the contrary: interior of Koch snowflake admissible.



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Recall square function

$$Sf(x) = \left( \int_0^\infty \fint_{|y-x|<t} |t\partial_t e^{-t\sqrt{L}} f|^2 \frac{dydt}{t} \right)^{\frac{1}{2}}.$$

But for $x \in O_1$ square function $Sf(x)$ uses values on $O_2$!?

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Let $a\colon O \to \mathbb{C}$ measurable, $B$ ball.

**Definition**

Call $a$ usual atom if $supp(a) \subseteq B$, $\|a\|_2 \leq |B \cap O|^{-\frac{1}{2}}$, $\int_O a \, dx = 0$.

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Call $a$ atom near $D$ if $B$ near $D$, $supp(a) \subseteq B$, $\|a\|_2 \leq |B \cap O|^{-\frac{1}{2}}$.

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Note: theory for $H^1_D$ thus unifies pure Dirichlet/Neumann cases!

# Strategy from Auscher–Russ (Lipschitz)

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Tasks/ideas:

- Eliminate "detour" via maximal space (uses Lipschitz in Green's formula)!
- Develop duality theory for $H^1_D(O)$ from scratch!

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Kernel bounds + some general theory for mixed BC: recent paper by Böhnlein–Ciani–Egert (to appear in Math. Ann.).

# Duality theory for $H^1_D(O)$

Matching $BMO$-space:

$$\|f\|_{BMO_D} = \sup_{B \text{ usual}} \left( \fint_B |f - (f)_B|^2 \, dx \right)^{\frac{1}{2}} + \sup_{B \text{ near } D} \left( \int_B |f|^2 \, dx \right)^{\frac{1}{2}} .$$

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**Theorem**

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- Important observation: $\varphi \in C_c(O)$, $B$ ball near $D \implies B$ needs minimal size (depending on $\varphi$) to hit $\operatorname{supp}(\varphi)$.

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**Observation**

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Why is $H^1_L$ complete? Not at all clear from definition! But it is dual of $VMO_D$!

# Application to the Laplacian

**Corollary**

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Then

$$H^1_{-\Delta_D} = H^1_D = H^1_{-\Delta_{D,max}}.$$

Thank you for your attention!

A digital version of this presentation can be found here: