Wellposedness of non-linear evolution equations driven by rough coefficients

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(joint work: Pascal Auscher)

Harmonic Analysis & Fluid Flows

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Non-linear model problem

Let $\Phi : \mathbb{R} \to \mathbb{R}^n$, $\quad |\Phi(u)| \lesssim |u|^{1+\delta}$, $\quad |\Phi(u) - \Phi(v)| \lesssim (|u|^{\delta} + |v|^{\delta})|u-v|$.

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Let $\Phi : \mathbb{R} \to \mathbb{R}^n$, $\quad |\Phi(u)| \lesssim |u|^{1+\rho}$, $\quad |\Phi(u) - \Phi(v)| \lesssim (|u|^{\rho} + |v|^{\rho})\,|u-v|$.

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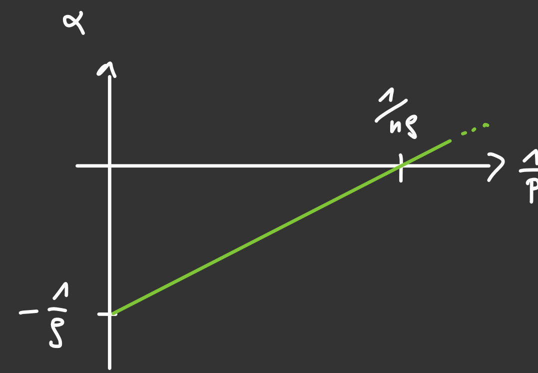
Which initial data?



Scaling argument suggests:

$$u_0 \in \dot{B}^{\alpha}_{p,p} \qquad \text{with} \quad \frac{1}{\rho} = \frac{n}{p} - \alpha$$

"critical spaces"

Weak solutions

Def: Call u weak solution of

$$\partial_t u - \mathrm{div}(A\nabla u) = \mathrm{div}(\phi(u)), \quad u(0) = u_0$$

if:

(a) $u, \nabla u, \phi(u) \in L^2_{loc}(0,T; L^2_{loc}(\mathbb{R}^n))$

(b) for all $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^n)$:

$$\iint -u\,\partial_t\varphi + A\nabla u\cdot\nabla\varphi = -\iint \phi(u)\cdot\nabla\varphi .$$

(c) $u(t) \to u_0$ in $\mathcal{D}'$ as $t \to 0$.

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Rem: Definition does not use operator theory!

Besov & Z-spaces

Let $p \in (1,\infty]$, $\alpha < 0$. Fix $u_0 \in \dot{B}^{\alpha}_{p,p}$. Set $L := -\mathrm{div}(A\nabla)$.

$$\boxed{\mathcal{E}_L(u_0)(t) := e^{-tL} u_0 \in \,?}$$

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Def.: Let $q, r \in (1,\infty]$, $\beta \in \mathbb{R}$. Define $Z^{r,q}_{\beta}$ via

$$f \in Z^{r,q}_{\beta} \iff \left(\fint_{t/2}^{t} \fint_{B(x,\sqrt{t})} |s^{-\beta} f(s,y)|^q \, dy \, ds \right)^{\frac{1}{q}} \in L^r\left(\mathbb{R}^{n+1}_+, \frac{dx\,dt}{t}\right)$$

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Example: $Z^{r,r}_{\beta} = L^r_{\beta}$ by Fubini.

Besov & Z-spaces (continued)

Still $u_0 \in \dot{B}^{\alpha}_{p,p}$. Put $\delta := \frac{n}{p} - \alpha$.

$$\Rightarrow \quad \mathcal{E}_L(u_0) \in Z^{p,q}_{\frac{\alpha}{2}} = Z^{p,q}_{\frac{n}{2p} - \frac{\delta}{2}} \qquad \text{for any } q \in (1,\infty].$$

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Still $u_0 \in \dot{B}^{\alpha}_{p,p}$. Put $\sigma := \frac{n}{p} - \alpha$.

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Embeddings:

- $Z^{r, q_0}_{\frac{n}{2r} + \beta} \subseteq Z^{r, q_1}_{\frac{n}{2r} + \beta}$ if $q_0 \geq q_1$
- $Z^{r_0, q}_{\frac{n}{2r_0} + \beta} \subseteq Z^{r_1, q}_{\frac{n}{2r_1} + \beta}$ if $r_0 \leq r_1$ ("Sobolev embedding")

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Prop: $u_0 \in \dot{B}^{\alpha}_{p,p} \Rightarrow \mathcal{E}_L(u_0) \in Z^{r,q}_{\frac{n}{2r} - \frac{\delta}{2}}$ for any $r \geq p$, $q \in (1, \infty]$.

Wellposedness result (BE) (simplified version)

Set $BE\_(n,s) := s(n+2)$.

Thm: (Anscher – B.) Let $s>1$, $p\in(1,\infty]$ and $\alpha\in(-1,0)$ satisfying $\frac{1}{s}=\frac{n}{p}-\alpha$.

Fix $u_0\in\dot{B}^{\alpha}_{p,p}$ if $p<\infty$ and $u_0\in{}_v\dot{B}^{\alpha}_{\infty,\infty}$ if $p=\infty$.

Let $r,q>BE\_(n,s)$ with $r\geq p$.

$\Rightarrow$ exists maximal, unique, weak solution u to (BE) satisfying $u\in Z^{r,q}_{\frac{n}{2r}-\frac{1}{2s}}$ and $\nabla u\in Z^{\frac{r}{1+s},2}_{\frac{n(1+s)}{r}-\frac{1}{2s}-\frac{1}{2}}$

Moreover, $u\in Z^{\tilde{r},\tilde{q}}_{\frac{n}{2\tilde{r}}-\frac{1}{2s}}$ for any $\tilde{q}\in(1,\infty]$ and $\tilde{r}>BE\_(n,s)$.

Mild solutions & existence

$$R^L_{\frac{1}{2}}(F)(t) := \int_0^t e^{-(t-s)L} \operatorname{div}(F(s))\, ds \qquad \text{"formally"}$$

Extend to $R^L_{\frac{1}{2}} : Y \to X$, $\quad X, Y$ some Z-spaces

Def: $u \in X$ called mild solution if $\quad u = \mathcal{E}_L(u_0) + R^L_{\frac{1}{2}}(\Phi(u))$.

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Set $X := L^r_{\frac{n}{2r} - \frac{1}{2s}}$. $\qquad u \in X \Rightarrow \Phi(u) \in L^{r/1+s}_{\frac{n(1+s)}{2r} - \frac{1}{2s} - \frac{1}{2}} =: Y$

Q: $R^L_{\frac{1}{2}}(\Phi(u)) \in X$?

Mild solutions & existence

$$R^L_{\frac12}(F)(t) := \int_0^t e^{-(t-s)L} \operatorname{div}(F(s))\,ds \qquad \text{"formally"}$$

Extend to $R^L_{\frac12} : Y \to X$, $\quad X, Y$ some Z-spaces

Def: $u \in X$ called mild solution if $\quad u = \mathcal{E}_L(u_0) + R^L_{\frac12}(\Phi(u))$.

Set $X := L^r_{\frac{n}{2r} - \frac{1}{2s}}$. $\quad u \in X \Rightarrow \Phi(u) \in L^{r/1+s}_{\frac{n(1+s)}{2r} - \frac{1}{2s} - \frac12} =: Y$

Q: $R^L_{\frac12}(\Phi(u)) \in X$? $\quad$ Yes, if $BE\_(n,s) \leq r$.

Then: fixed-point argument works :-)

What regularity?

Some concerns regarding mild solution:

- fixed-point space $L^r_{\frac{n}{2r}-\frac{1}{2s}}$ artificial...

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Prop: $u \in L^{r}_{\frac{n}{2r}-\frac{1}{2s}}$ mild solution $\Longrightarrow$ $u \in Z^{\tilde{r},\tilde{q}}_{\frac{n}{2\tilde{r}}-\frac{1}{2s}}$, $\tilde{r} > BE_-(n,s)$, $\tilde{r} \geq p$, any $q \in (1,\infty]$.

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Proof: Stay tuned...

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Cor.: (Existence weak) Ex. $u \in Z^{r,q}_{\frac{n}{2r} - \frac{1}{2s}}$ weak sol. with $r > BE\_(n,s)$, $r \geq p$.

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Proof: Let $u \in L^{r}_{\frac{n}{2r}-\frac{1}{2s}}$ mild solution.

Bootstrapping $\Rightarrow$ $u \in Z^{r,2(1+s)}_{\frac{n}{2r}-\frac{1}{2s}}$

$\Rightarrow$ $\phi(u) \in Z^{\hat{r},2}_{\frac{n}{2\hat{r}}-\frac{1}{2s}-\frac{1}{2}}$, where $\hat{r} := \frac{r}{1+s} \geq 2$.

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Auscher–Hou with $F := \phi(u)$

$\Rightarrow$ u weak solution (of "linear" equation).

"Hypercontractive" SIOs

Standard case: Let $p \geq q$, $\beta > -1$.

$$S : L^q_\beta \to L^q_{\beta+k} + \text{kernel} + \text{decay} \implies S : Z^{p,q}_\beta \to Z^{p,q}_{\beta+k}$$

Ex.: $S = R^L_{1/2}$, $k = 1/2$, $q = 2$.

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parabolic

Used for mild existence: $R^L_{\frac{1}{2}} : L^q_\beta \to L^r_{\beta + \frac{1}{2} - \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}$, $q \leq r \leq q^*$

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Standard case: Let $p \geq q$, $\beta > -1$.

$$S : L^q_\beta \to L^q_{\beta+\kappa} \; + \text{Kernel} + \text{decay} \;\Rightarrow\; S : Z^{p,q}_\beta \to Z^{p,q}_{\beta+\kappa}$$

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$$S : L^q_\beta \to L^r_{\beta+\kappa-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \; + \text{Kernel} + \text{decay} \;\Rightarrow\; S : Z^{p,q}_\beta \to Z^{p,r}_{\beta+\kappa}.$$

Rem.: $S : L^q_\beta \to Z^{q,r}_{\frac{n}{2q}+\beta+\kappa-\frac{n}{2q}} \subseteq L^r_{\beta+\kappa-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}$ (factorization)

Bootstrapping (continued)

Prop: $u \in L^{r}_{\frac{n}{2r}-\frac{1}{2s}}$ mild solution $\Rightarrow$ $u \in Z^{\tilde{r},\tilde{q}}_{\frac{n}{2\tilde{r}}-\frac{1}{2s}}$, $\tilde{r} > BE\_(n,s)$, $\tilde{r} \geq p$, any $q \in (1,\infty]$.

Sketch of proof:

Reg. of u $\Rightarrow$ $\phi(u) \in L^{r/1+s}_{\frac{n(1+s)}{2r}-\frac{1}{2s}-\frac{1}{2}}$

$\Rightarrow$ $R^{L}_{1/2}(\phi(u)) \in Z^{r/1+s,\, r}_{\frac{n(1+s)}{2r}-\frac{1}{2s}} =: \tilde{X}$

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Hence, $u = \mathcal{E}_L(u_0) + R^{L}_{1/2}(\phi(u)) \in \tilde X$

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Iteration $\Rightarrow$ Can lower to $\tilde{r} > BE\_(n,s)$, $\tilde{r} \geq p$.

Uniqueness weak solution

Summary so far: Ex. weak solution with desired regularity.

Idea uniqueness: u weak solution $\overset{?}{\Longrightarrow}$ u mild solution

Then: at most one mild solution classical

uniqueness weak solution (continued)

Sketch of proof: (weak $\Rightarrow$ mild)

Let u weak solution with $\nabla u \in Z^{\hat{r},2}_{\frac{n}{2\hat{r}} - \frac{1}{2s} - \frac{1}{2}}$.

Put $v := \mathcal{E}_L(u_0) + R^L_{\frac{1}{2}}(\phi(u))$

uniqueness weak solution (continued)

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$\Rightarrow$ v weak solution $\partial_t v - \mathrm{div}(A\nabla v) = \mathrm{div}(\phi(u))$, $v(0) = u_0$

$\Rightarrow$ $w := u - v$ solves $\partial_t w - \mathrm{div}(A\nabla w) = 0$, $w(0) = 0$, $\nabla w \in Z^{\hat{r},2}_{\frac{n}{2\hat{r}}-\frac{1}{2s}-\frac{1}{2}}$.

uniqueness weak solution (continued)

Sketch of proof: (weak $\Rightarrow$ mild)

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Linear uniqueness [Auscher, Hou] $\Rightarrow$ $w = 0$ $\Rightarrow$ $u = v$.

Thank you for your attention!