The Kato square root problem on irregular open sets

Sebastian Bechtel
TU Darmstadt

TECHNISCHE UNIVERSITÄT DARMSTADT

Parabolic Evolution Equations, Harmonic Analysis and Spectral Theory

Bad Herrenalb, 8 May 2019

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- $A \in \mathrm{L}^\infty(O; \mathbb{C}^{d \times d})$
- define sesquilinear form

$$a(u, v) := \int_O A\nabla u \cdot \overline{\nabla v} dx \qquad (u, v \in \mathcal{V})$$

- A coercive in Gårding's sense

$$Re\, a(u, u) \gtrsim \|\nabla u\|^2_{\mathrm{L}^2(O)} \qquad (u \in \mathcal{V})$$

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Problem

For which spaces $\mathcal{V}$ do we have $\mathrm{D}(L^{\frac{1}{2}}) = \mathcal{V}$ with equivalent norms?

What is known for mixed boundary conditions?

Theorem (Egert, Haller-Dintelmann, Tolksdorf ’14 & ’16)

Suppose:

- *O bounded domain*
- *O is d-regular*
- *∂O is $(d-1)$-regular.*
- *$D \subseteq \partial O$ is $(d-1)$-regular*
- *$\overline{\partial O \setminus D}$ admits bi-Lipschitz charts*

Then the Kato property holds for $\mathcal{V} = \mathrm{H}^{1,2}_{D}(O)$.

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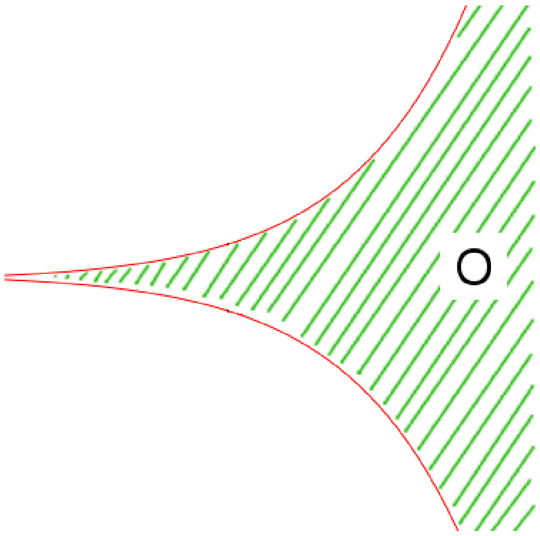
- localization and thickening of *O*: no *d*-regularity

Thickening of O

For simplicity: pure Dirichlet boundary conditions

- For example: O is an unbounded cusp domain $\rightsquigarrow$ not d-regular



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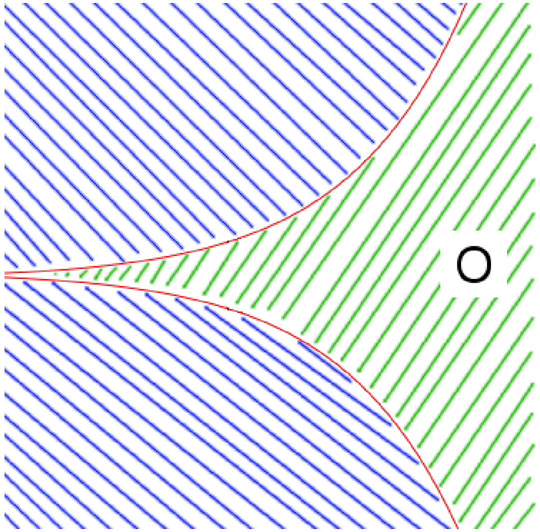

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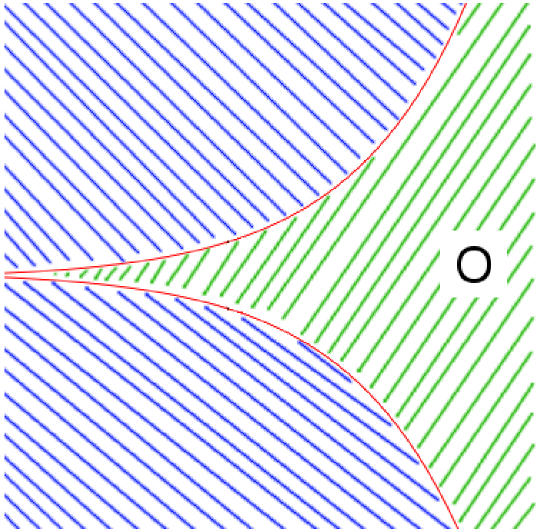

Question

How do the Kato problems on O and $\mathbf{O}$ relate?

Idea: relate functional calculi of L and $\mathbf{L}$

“Localization” of the functional calculus

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1. $\mathcal{Q}\mathbf{L} \subseteq \mathbf{L}\mathcal{Q}$ for *good* projection $\mathcal{Q}$

Calculate with *good* projection $\mathcal{Q}$ and $u \in \mathrm{D}(\mathcal{Q}\mathbf{L}) = \mathrm{D}(\mathbf{L})$:

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hence: $\mathcal{Q}u \in \mathrm{D}(\mathbf{L})$ and $\mathbf{L}\mathcal{Q}u = \mathcal{Q}\mathbf{L}u$

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2. decomposition of functional calculus and operator domains

- $\mathcal{Q}_1$ good projection
- $\mathbf{L}_1$ and $\mathbf{L}_2$ the restrictions of $\mathbf{L}$ to $\mathcal{Q}_1\mathrm{L}^2(\mathbf{O})$ and $\underbrace{(1-\mathcal{Q}_1)}_{=\mathcal{Q}_2}\mathrm{L}^2(\mathbf{O})$

Then

$$u \in \mathrm{D}(f(\mathbf{L})) \quad \Longleftrightarrow \quad \mathcal{Q}_1 u \in \mathrm{D}(f(\mathbf{L}_1)) \text{ and } \mathcal{Q}_2 u \in \mathrm{D}(f(\mathbf{L}_2))$$

with

$$f(\mathbf{L})u = f(\mathbf{L}_1)\mathcal{Q}_1 u + f(\mathbf{L}_2)\mathcal{Q}_2 u.$$

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3. $\mathcal{Q}_1 = \mathbb{1}_O$ is a *good* projection

- multiplication operators commute with each other

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- $\nabla \mathcal{Q} \varphi = \mathcal{Q} \nabla \varphi$ for $\varphi \in \mathrm{C}_0^\infty(\mathbf{O})$

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- multiplication operators commute with each other
- $\nabla \mathcal{Q}\varphi = \mathcal{Q}\nabla\varphi$ for $\varphi \in \mathrm{C}_0^\infty(\mathbf{O})$
- $\nabla \mathcal{Q} = \mathcal{Q}\nabla$ on $\mathrm{H}_0^{1,2}(\mathbf{O})$ by density

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4. putting it all together

Kato for $\mathbf{L}$ implies

$$\mathcal{Q}_1 \mathrm{H}_0^{1,2}(\mathbf{O}) = \mathcal{Q}_1 \mathrm{D}(\mathbf{L}^{\frac{1}{2}}) = \mathrm{D}(\mathbf{L}_1^{\frac{1}{2}})$$

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and for $u \in \mathrm{D}(\mathbf{L}_1^{\frac{1}{2}})$ we get

$$\|\mathbf{L}_1^{\frac{1}{2}} u\|_{\mathrm{L}^2} = \|\mathbf{L}^{\frac{1}{2}} u\|_{\mathrm{L}^2} \approx \|u\|_{\mathrm{H}_0^{1,2}}$$

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identify: $\mathrm{L}^2(O) \sim \mathcal{Q}_1 \mathrm{L}^2(\mathbf{O})$ and $\mathrm{H}_0^{1,2}(O) \sim \mathcal{Q}_1 \mathrm{H}_0^{1,2}(\mathbf{O})$

$$\rightsquigarrow L = \mathbf{L}_1$$

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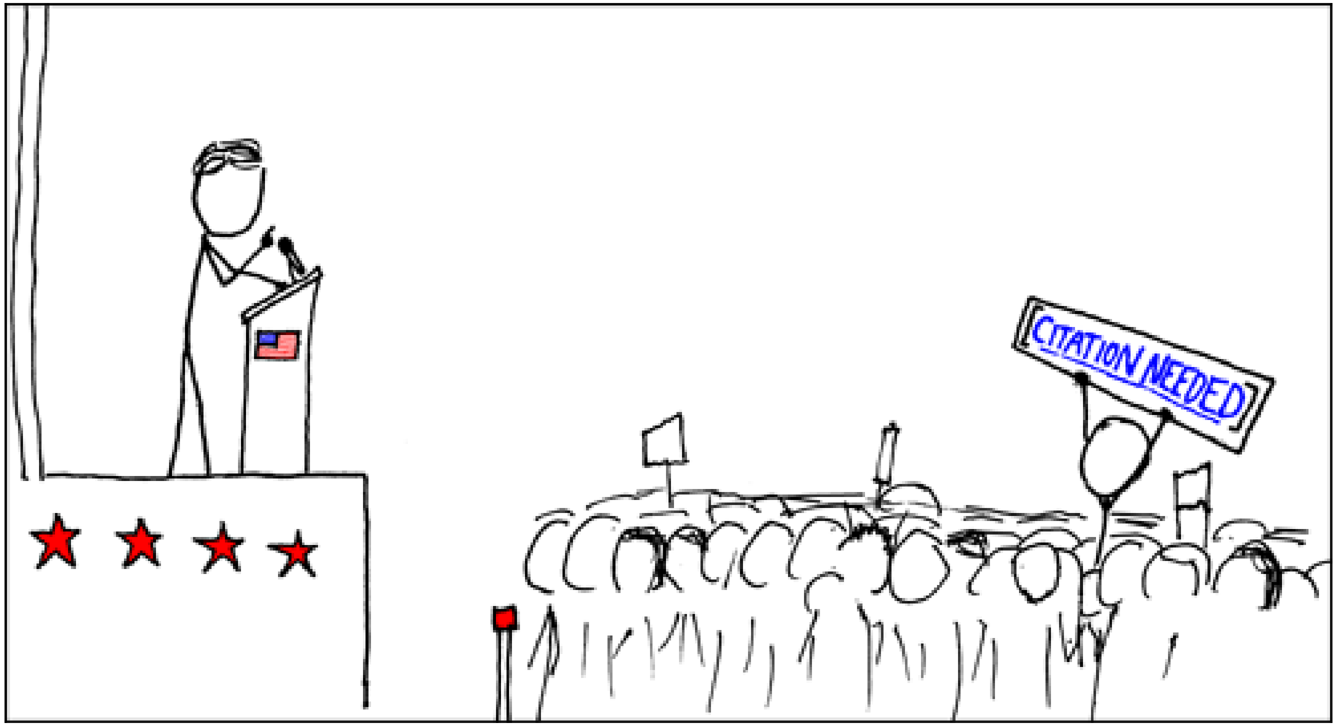
Now, it's time for conference dinner!



S. Bechtel, R. Haller-Dintelmann. *The Kato square root problem on irregular open sets*. Available on arXiv.