Wellposedness of non-linear evolution equations driven by rough coefficients

Sebastian Bechtel

Université Paris-Saclay

(joint work: Pascal Auscher)

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Non-linear model problem

Let $\Phi : \mathbb{R} \to \mathbb{R}^n$, $\quad |\Phi(u)| \lesssim |u|^{1+\rho}$, $\quad |\Phi(u) - \Phi(v)| \lesssim (|u|^\rho + |v|^\rho)|u-v|$.

Reaction-Diffusion problem:

$$\partial_t u - \operatorname{div}(A\nabla u) = \Phi(u), \quad u(0) = u_0. \qquad \text{(RD)}$$

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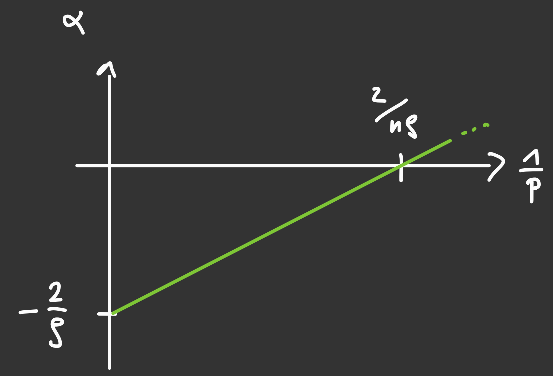
Which initial data?

Scaling argument suggests:

$$u_0 \in \dot{B}^{\alpha}_{p,p} \qquad \text{with} \quad \frac{2}{\rho} = \frac{n}{p} - \alpha$$

"critical spaces"



Weak solutions

Def: Call (u,T) <u>weak solution</u> of

$$\partial_t u - \operatorname{div}(A\nabla u) = \phi(u), \quad u(0) = u_0$$

if:

(a) $u, \nabla u \in L^2_{loc}(0,T; L^2_{loc}(\mathbb{R}^n))$, $\phi(u) \in L^{2_*}_{loc}(0,T; L^{2_*}_{loc}(\mathbb{R}^n))$

(b) for all $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^n)$:

$$\iint -u\,\partial_t\varphi + A\nabla u\cdot\nabla\varphi = \iint \phi(u)\,\varphi.$$

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(c) $u(t) \to u_0$ in $\mathcal{D}'$ as $t \to 0$.

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Rem: Definition does <u>not</u> use operator theory!

Besov & $\mathcal{Z}$-spaces

Let $p \in (1, \infty]$, $\alpha < 0$. Fix $u_0 \in \dot{B}^{\alpha}_{p,p}$. Set $L := -\operatorname{div}(A\nabla)$.

$$\boxed{\mathcal{E}_L(u_0)(t) := e^{-tL} u_0 \in ?}$$

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<u>Def.</u>: Let $q, r \in (1,\infty]$, $\beta \in \mathbb{R}$. Define $Z^{r,q}_{\beta}$ via

$$f \in Z^{r,q}_{\beta} \iff \left(\fint_{t/2}^{t} \fint_{B(x,\sqrt{t})} |s^{-\beta} f(s,y)|^q \, dy \, ds \right)^{\frac{1}{q}} \in L^r\left(\mathbb{R}^{n+1}_+, \frac{dx\,dt}{t}\right)$$

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Example: $Z^{r,r}_{\beta} = L^{r}_{\beta}$ by Fubini.

Besov & Z-spaces (continued)

Still $u_0 \in \dot{B}^{\alpha}_{p,p}$. Put $\sigma := \frac{n}{p} - \alpha$.

$$\Rightarrow \quad \mathcal{E}_L(u_0) \in Z^{p,q}_{\frac{\alpha}{2}} = Z^{p,q}_{\frac{n}{2p} - \frac{\sigma}{2}} \qquad \text{for any } q \in (1, \infty].$$

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Embeddings:

- $Z^{r, q_0}_{\frac{n}{2r} + \beta} \subseteq Z^{r, q_1}_{\frac{n}{2r} + \beta}$ if $q_0 \geq q_1$
- $Z^{r_0, q}_{\frac{n}{2r_0} + \beta} \subseteq Z^{r_1, q}_{\frac{n}{2r_1} + \beta}$ if $r_0 \leq r_1$ ("Sobolev embedding")

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Prop: $u_0 \in \dot{B}^{\alpha}_{p,p} \Rightarrow \mathcal{E}_L(u_0) \in Z^{r,q}_{\frac{n}{2r} - \frac{\delta}{2}}$ for any $r \geq p$, $q \in (1, \infty]$.

Wellposedness result (RD)

Set $RD_-(n,s) := \frac{s}{2}(n+2)$, $RD_+(n,s) := n(1+s)^{s/2}$.

<u>Thm:</u> (Anscher – B.) Let $s > \frac{2}{n}$, $p \in (1, RD_+(n,s))$, and set $\alpha := \frac{n}{p} - \frac{2}{s}$.

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- (Existence) There exists weak solution (u,τ) of (RD).
- (Regularity) For any $r \in (RD_-(n,s), RD_+(n,s))$ with $r \geq p$,

  $q > RD_-(n,s)$: $u \in Z^{r,q}_{\frac{n}{2r}-\frac{1}{s}}(T)$, $\nabla u \in Z^{r,2}_{\frac{n}{2r}-\frac{1}{s}-\frac{1}{2}}(T)$ $\forall T \in (0,\tau)$.

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- (Maximality) Solution (u,τ) maximal in all uniqueness classes.

Mild solutions & existence

$$\mathcal{L}_1^L(f)(t) := \int_0^t e^{-(t-s)L} f(s)\, ds \qquad \text{"formally"}$$

Extend to $\mathcal{L}_1^L : Y \to X$, $\quad X, Y$ some Z-spaces

Def: $u \in X$ called mild solution if $\quad u = \mathcal{E}_L(u_0) + \mathcal{L}_1^L(\Phi(u))$.

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Set $X := L^r_{\frac{n}{2r} - \frac{1}{3}}$. $\qquad u \in X \implies \Phi(u) \in L^{r/1+s}_{\frac{n(1+s)}{2r} - \frac{1}{3} - \frac{1}{2}} =: Y$

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Then: fixed-point argument works :-)

What regularity?

Some concerns regarding mild solution:

- fixed-point space $L^r_{\frac{n}{2r}-\frac{1}{3}}$ artificial...

- enough (local) integrability for mild solution?!

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Prop: $u \in L^{r}_{\frac{n}{2r}-\frac{1}{s}}$ mild solution of (RD)

$$\Rightarrow \quad u \in Z^{\tilde{r},\tilde{q}}_{\frac{n}{2\tilde{r}}-\frac{1}{s}}, \quad \tilde{r} \in (RD_-(n,s), RD_+(n,s)), \ \tilde{r} \geq p, \ q \in (1,\infty].$$

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Proof: Stay tuned...

Existence weak solution

Cor.: (Existence weak) Ex. $u \in L^{r}_{\frac{n}{2r}-\frac{1}{s}}$ weak sol. with $r \in (RD_-(n,s), RD_+(n,s))$, $r \geq p$.

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Cor.: (Existence weak) Ex. $u \in L^r_{\frac{n}{2r}-\frac{1}{\varrho}}$ weak sol. with $r \in (RD_-(n,\varrho), RD_+(n,\varrho))$, $r \geq p$.

Proof: Let $u \in L^r_{\frac{n}{2r}-\frac{1}{\varrho}}$ mild solution.

Bootstrapping $\Rightarrow$ $u \in Z^{r,2(1+\varrho)}_{\frac{n}{2r}-\frac{1}{\varrho}}$

$\Rightarrow$ $\phi(u) \in Z^{\hat{r},2}_{\frac{n}{2\hat{r}}-\frac{1}{\varrho}-1}$, where $\hat{r} := \frac{r}{1+\varrho} > 1$.

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Auscher–Hou with $F := \phi(u)$

$\Rightarrow$ u weak solution (of "linear" equation).

"Hypercontractive" SIOs

Standard case: Let $p \geq q$, $\beta > -1$.

$$S : L^q_\beta \to L^q_{\beta+k} + \text{kernel} + \text{decay} \implies S : Z^{p,q}_\beta \to Z^{p,q}_{\beta+k}$$

Ex.: $S = \mathcal{L}^L_1$, $k = 1$, $q = 2$.

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parabolic

Used for mild existence: $\mathcal{L}^L_1 : L^q_\beta \to L^r_{\beta+1-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}$, $q \leq r \leq q^{**}$

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"Hypercontractive" SIOs

Standard case: Let $p \geq q$, $\beta > -1$.

$$S : L^q_\beta \to L^q_{\beta+\kappa} + \text{kernel} + \text{decay} \implies S : Z^{p,q}_\beta \to Z^{p,q}_{\beta+\kappa}$$

$q \in (1,\infty]$

Ex.: $S = \mathcal{L}^L_1$, $\kappa = 1$, $\cancel{q=2}$.

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Used for mild existence: $\mathcal{L}^L_1 : L^q_\beta \to L^r_{\beta+1-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}$, $q \leq r \leq q^{**}$

Thm.: Let $p \geq q$, $\beta > -1$.

$$S : L^q \to L^r_{\kappa-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})} + \text{kernel} + \text{decay} \implies S : Z^{p,q}_\beta \to Z^{p,r}_{\beta+\kappa}.$$

Rem.: $S : L^q_\beta \to Z^{q,r}_{\frac{n}{2q}+\beta+\kappa-\frac{n}{2q}} \subseteq L^r_{\beta+\kappa-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}$ (factorization)

Bootstrapping (continued)

Prop: $u \in L^{r}_{\frac{n}{2r}-\frac{1}{s}}$ mild solution $\Rightarrow$ $u \in Z^{\tilde r, \tilde q}_{\frac{n}{2\tilde r}-\frac{1}{s}}$, $\tilde r \in (RD\_(n,s), \infty]$,

$\tilde r \geq p$, $q \in (1, \infty]$.

Sketch of proof:

Reg. of u $\Rightarrow$ $\phi(u) \in L^{r/1+s}_{\frac{n(1+s)}{2r}-\frac{1}{s}-1}$

$$\Rightarrow \mathcal{L}^{L}_{1}(\phi(u)) \in L^{r/1+s}_{\frac{n(1+s)}{2r}-\frac{1}{s}} =: \tilde X$$

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Hence, $u = \mathcal{E}_L(u_0) + \mathcal{L}^{L}_{1}(\phi(u)) \in \tilde{X}$

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Hence, $u = \mathcal{E}_L(u_0) + \mathcal{L}^{L}_{1}(\phi(u)) \in \tilde{X}$

Iteration $\Rightarrow$ Can lower to $\tilde{r} > RD\_(n,s)$, $\tilde{r} \geq p$.

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Summary so far: Ex. weak solution with desired regularity.

Idea uniqueness: u weak solution $\overset{?}{\Longrightarrow}$ u mild solution

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<u>Idea uniqueness</u>: u weak solution $\overset{?}{\Longrightarrow}$ u mild solution

Then: at most one mild solution "classical"

uniqueness weak solution (continued)

Sketch of proof: (weak $\Rightarrow$ mild)

Let u weak solution (with $\nabla u \in Z^{r,2}_{\frac{n}{2r} - \frac{1}{s} - \frac{1}{2}}$).

Put $v := \mathcal{E}_L(u_0) + \mathcal{L}_1^L(\phi(u))$

Uniqueness weak solution (continued)

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Put $v := \mathcal{E}_L(u_0) + \mathcal{L}_1^L(\phi(u))$

$\Rightarrow$ v weak solution $\partial_t v - \operatorname{div}(A\nabla v) = \phi(u)$, $v(0) = u_0$

$\Rightarrow$ $w := u - v$ solves $\partial_t w - \operatorname{div}(A\nabla w) = 0$, $w(0) = 0$, $\nabla w \in Z^{r,2}_{\frac{n}{2r}-\frac{1}{s}-\frac{1}{2}}$.

uniqueness weak solution (continued)

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Linear uniqueness [Auscher, Hou] $\Rightarrow$ $w = 0$ $\Rightarrow$ $u = v$.

Thank you for your attention!