Wellposedness of linear & non-linear parabolic equations driven by non-smooth coefficients

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A linear starter...

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div}(F) & \text{on } (0,T)\times\mathbb{R}^n \\ u(0) = u_0 & \text{on } \mathbb{R}^n \end{cases}$$

$A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ measurable, bounded and elliptic:

$$|A(x)\xi| \leq \Lambda |\xi|, \qquad \operatorname{Re}(A(x)\xi \,|\, \xi) \geq \lambda |\xi|^2 \qquad (x \in \mathbb{R}^n, \xi \in \mathbb{C}^n)$$

<u>Important</u>: A has <u>no smoothness</u>

Which solutions? $\to$ classical

$\to$ "mild" solution

$\to$ weak solution

A cautionary tale

IVP: $\quad \partial_t u \underbrace{- \operatorname{div}(A \nabla u)}_{=: Lu} = 0, \quad u(0) = u_0.$

Solution (formally): $\quad u(t) = \mathcal{E}_L(u_0)(t) := e^{-tL} u_0$

"Counterexample" [Frehse]:

$\forall p > 2 \ \exists$ coefficients A: $\quad \nabla u(t) \notin L^p(\mathbb{R}^n)$

$\Rightarrow$ Be careful with operator theory $\rightsquigarrow$ mild solutions

double-edged sword

...

Weak solutions

Def: Call $u \in L^2_{loc}(0,T; W^{1,2}_{loc}(\mathbb{R}^n))$ <u>weak solution</u> of

$$\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div}(F), \quad u(0) = u_0,$$

distributions

if:

(a) for all $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^n)$:

$$\iint -u\,\partial_t\varphi + A\nabla u\cdot\nabla\varphi = -\iint F\cdot\nabla\varphi.$$

(b) $u(t) \to u_0$ in S' as $t \to 0$.

Rem: Definition does <u>not</u> use operator theory!

Which function spaces?

guiding principle: initial value problem

$$u_0 \in \begin{cases} \dot{B}^{\alpha}_{p,p} & \text{homogeneous Besov spaces} \\ \dot{H}^{\alpha,p} & \text{homogeneous "Hardy-Sobolev" spaces} \end{cases}$$ ← first

Let $u_0 \in \dot{H}^{\alpha,p}$, $\alpha < 0$, $p \in (1,\infty]$

$\Rightarrow \mathcal{E}_L(u_0) \in T^{p,2}_{\frac{\alpha}{2}}$ "weighted parabolic tent space"

$p = \infty$: $g \in T^{\infty,2}_{\beta}$

$$\Leftrightarrow \sup_{t,x} \left(\int_0^t \fint_{B(x,\sqrt{t})} |s^{-\beta} g(s,y)|^2 \frac{dy\,ds}{s} \right)^{\frac{1}{2}} < \infty$$

Origin: Coifman–Meyer–Stein '85 ⇝ unweighted & elliptic

Include forcing term F

Formally: $u(t) = R^L_{\frac{1}{2}}(F)(t) := \int_0^t e^{-(t-s)L} \operatorname{div}(F(s))\, ds$ solves

$$\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div}(F), \quad u(0).$$

Indeed, for $\alpha > -1$:

(a) Calderón-Zygmund theory:

$$F \in T^{p,2}_{\frac{\alpha}{2}-\frac{1}{2}} \implies u \in T^{p,2}, \quad \nabla u \in T^{p,2}_{\frac{\alpha}{2}-\frac{1}{2}}$$

(b) Approximation argument + L^2-theory (Lions)

$\implies$ u weak solution

(c) tedious work $\implies$ $u(0) = 0$.

What about uniqueness?

Suffices: $\partial_t u - \operatorname{div}(A\nabla u) = 0, \quad u(0) = 0 \quad \Rightarrow \quad u = 0.$

Even for heat equation: weak solutions <u>not</u> unique

<u>Thm:</u> Let $\beta > -1$. Then $\nabla u \in T^{p,2}_{\beta}$ sufficient for uniqueness.

Idea of proof:

(1) for $0 < s < t$, $\phi \in C_c^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} u(t)\, \phi \, dx = \int_{\mathbb{R}^n} u(s)\, e^{-(t-s)L^*} \phi \, dx \qquad \text{(homotopy identity).}$$

(2) Take limit $s \to 0 \Rightarrow \text{RHS} = 0.$

Summary so far

Thm: (Auscher – Hou) Let $\alpha \in (-1, 0)$, $p \in (1, \infty]$, $u_0 \in \dot{H}^{\alpha, p}$, $F \in T^{p,2}_{\frac{\alpha}{2} - \frac{1}{2}}$.

Then:

$$\partial_t u - \operatorname{div}(A \nabla u) = \operatorname{div}(F), \quad u(0) = u_0$$

admits weak solution $u \in T^{p,2}_{\frac{\alpha}{2}}$ satisfying $\nabla u \in T^{p,2}_{\frac{\alpha}{2} - \frac{1}{2}}$.

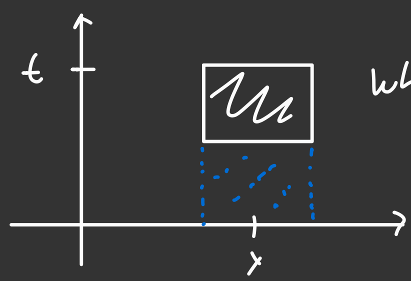
Moreover: u unique among all weak solutions v with $\nabla v \in T^{p,2}_{\frac{\alpha}{2} - \frac{1}{2}}$.

What about Besov data?

$u_0 \in \dot{B}^{\alpha}_{p,p}$, $\alpha < 0$, $p \in (1, \infty]$ $\Rightarrow$ $\mathcal{E}_L(u_0) \in Z^{p,2}_{\frac{\alpha}{2}}$ weighted Z-space

Go back to Barton–Mayboroda, Amenta ... Currently L. Haardt

$$g \in Z^{p,q}_{\beta} \iff \left(\int_0^\infty \int_{\mathbb{R}^n} \left| \left(\fint_{\frac{t}{2}}^{t} \fint_{B(x,\sqrt{t})} |s^{-\beta} g(s,y)|^q \, dy \, ds \right)^{\frac{1}{q}} \right|^p \frac{dx\, dt}{t} \right)^{\frac{1}{p}} < \infty.$$



Whitney box

Summary so far

Thm: (Auscher – Hou) Let $\alpha \in (-1,0)$, $p \in (1,\infty]$, $u_0 \in \overset{\dot B^{\alpha}_{p,p}}{\cancel{\dot H^{\alpha,p}}}$, $F \in \overset{Z}{\cancel{T}}{}^{p,2}_{\frac{\alpha}{2}-\frac{1}{2}}$.

Then:

$$\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div}(F), \quad u(0) = u_0$$

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Next: non-linear problems

Let $\Phi : \mathbb{R} \to \mathbb{R}^n$, $\quad |\Phi(u)| \lesssim |u|^{1+\beta}$, $\quad |\Phi(u) - \Phi(v)| \lesssim (|u|^\beta + |v|^\beta)|u-v|$.

Burger's type problem:

$$\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div}(\Phi(u)), \quad u(0) = u_0. \qquad \text{(BE)}$$

Which initial data?

Let $X \in \{B, H\}$. Scaling argument suggests

$$u_0 \in \dot{X}^{\alpha,p} \quad \text{with} \quad \frac{1}{\beta} = \frac{n}{p} - \alpha \qquad \text{"critical spaces"}$$

Comparison with Navier–Stokes

Replace $-\operatorname{div}(A\nabla\,\cdot)$ by Stokes operator $-\mathbb{P}\Delta$

Put $\Phi(u) = u \otimes u$.

(BE) with $\rho = 1$ $\leftrightsquigarrow$ mild formulation NS

Koch–Tataru: NS well-posed for $u_0 \in BMO^{-1}$ small.

One has $BMO^{-1} = \dot{H}^{-1,\infty}$. Also: $\frac{1}{1} = \frac{n}{\infty} - (-1)$

$\Rightarrow$ criticality condition for (BE)

Existence by fixed-point argument

Define: $\Theta(v) := \mathcal{E}_L(u_0) + R^L_{\frac{1}{2}}(\Phi(v))$

Fixed point Θ $\hat{=}$ "mild solution" of (BE).

Let $v \in Z^{p,2}_{\frac{\alpha}{2}} = Z^{p,2}_{\frac{n}{2p}-\frac{1}{2s}}$ $\Rightarrow$ $F := \Phi(v) \in Z^{p/1+s,\, 2/1+s}_{\frac{n(1+s)}{2p}-\frac{1}{2s}-\frac{1}{2}}$.

We know: $R^L_{\frac{1}{2}}(\Phi(v)) \in Z^{p/1+s,\, 2/1+s}_{\frac{n(1+s)}{2p}-\frac{1}{2s}} \overset{\text{embedding}}{\subseteq} Z^{p,\, 2/1+s}_{\frac{n}{2p}-\frac{1}{2s}}$ <u>no self-map!</u>

Hence, <u>need</u>: "hypercontractive" Calderón–Zygmund theory

Z-spaces: feasible $\qquad$ T-spaces: problematic

Wellposedness result (BE) (simplified version)

Set $BE\_(n,s) := s(n+2)$.

Thm: (Anscher – B.) Let $s > 1$, $p \in (1,\infty]$ and $\alpha \in (-1,0)$ satisfying $\frac{1}{s} = \frac{n}{p} - \alpha$.

Fix $u_0 \in \dot{B}^{\alpha}_{p,p}$ if $p < \infty$ and $u_0 \in {}_{v}\dot{B}^{\alpha}_{\infty,\infty}$ if $p = \infty$.

Let $r, q > BE\_(n,s)$ with $r \geq p$.

$\Rightarrow$ exists maximal, unique, weak solution u to (BE) satisfying $u \in Z^{r,q}_{\frac{n}{2r} - \frac{1}{2s}}$ and $\nabla u \in Z^{\frac{r}{1+s},2}_{\frac{n(1+s)}{r} - \frac{1}{2s} - \frac{1}{2}}$

Moreover, $u \in Z^{\tilde{r},\tilde{q}}_{\frac{n}{2\tilde{r}} - \frac{1}{2s}}$ for any $\tilde{q} \in (0,\infty]$ and $\tilde{r} > BE\_(n,s)$.

Some more non-linear problems

Reaction-diffusion:

$$\partial_t u - \operatorname{div}(A\nabla u) = \phi(u), \qquad u(0) = u_0.$$

Quasi-linear problem:

$$\partial_t u - \operatorname{div}(a(u)\nabla u) = 0, \qquad u(0) = u_0.$$

Lipschitz

BUC

Thank you for your attention!