Critical equations in an extended variational setting

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Workshop on Maximal Regularity & Related Topics

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Main result

$$du + Au\,dt = (F(u) + f)\,dt + (Bu + G(u) + g)\,dW(t)$$

$$+ \int_Z C(z)u(\cdot-) + H(u(\cdot-), z) + h(z)\,\tilde{N}(dz, dt)$$

$$u(0) = u_0.$$

"singular" lower order terms allowed

* local well-posedness $\longrightarrow$ $\boxed{\text{today's topic}}$
* blowup criteria
* global well-posedness with coercivity

model example (weak Allen–Cahn, $n=2$)

$$du - \Delta u \, dt = (\cancel{u} - u^3)\, dt + \text{transport noise}$$
$$u(0) = u_0 .$$

where $-\Delta : H^1_0(\mathcal{O}) \to H^{-1}(\mathcal{O})$, $\mathcal{O} \subseteq \mathbb{R}^2$ open
$u_0 \in L^2(\mathcal{O})$ a.s.

scaling considerations $\Longrightarrow$ $\boxed{\text{problem critical}}$

(ignore stochastics for a moment ...)

evolution equation perspective

$V := H^1_0(O)$, $H := L^2(O)$. $MR := C([0,T]; H) \cap L^2([0,T]; V)$.

fixed point map:

$$\Pi : v \in MR \underset{\boxed{?}}{\longmapsto} -v^3 \in L^2([0,T]; V^*) \longmapsto u \in MR$$

$\uparrow$

sol. of lin. prob.

$$du - \Delta u \, dt = -u^3 \, dt$$
$$u(0) = u_0$$

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(sub)criticality condition

sol. of lin. prob.

$$du - \Delta u \, dt = -u^3 \, dt$$
$$u(0) = u_0$$

(Sub) Criticality Condition

$X_\beta := [V^*, V]_\beta$, $\|\cdot\|_\beta := \|\cdot\|_{X_\beta}$

$\exists\, \delta \geq 0,\ \beta \in (1/2, 1)$: $\boxed{(1+\delta)(2\beta - 1) \leq 1}$ (*) and

$$\underbrace{\|F(u)\|_{V^*} \lesssim 1 + \|u\|_\beta^{1+\delta}}_{\text{growth condition}}, \quad \underbrace{\|F(u) - F(v)\|_{V^*}}_{\text{Lipschitz condition}} \lesssim (1 + \|u\|_\beta^\delta + \|v\|_\beta^\delta)\|u-v\|_\beta$$

Allen-Cahn:

$$\|F(u)\|_{L^1} = \|u\|_{L^3}^3 \lesssim \|u\|_\beta^{1+\delta}$$

with $\delta = 2$, $\beta = 2/3$

fullfils (*)

(Sub) Criticality Condition

$X_\beta := [V^*, V]_\beta$, $\|\cdot\|_\beta := \|\cdot\|_{X_\beta}$

$\exists\, \rho \geq 0,\ \beta \in (1/2, 1)$: $\boxed{(1+\rho)(2\beta - 1) \leq 1}$ $(*)$ and

$$\underbrace{\|F(u)\|_{V^*} \lesssim 1 + \|u\|_\beta^{1+\rho}}_{\text{growth condition}}, \quad \underbrace{\|F(u) - F(v)\|_{V^*} \lesssim (1 + \|u\|_\beta^\rho + \|v\|_\beta^\rho)\|u-v\|_\beta}_{\text{Lipschitz condition}}$$

Allen-Cahn: $\|F(u)\|_{V^*} \not\lesssim \|F(u)\|_{L^1} = \|u\|_{L^3}^3 \lesssim \|u\|_\beta^{1+\rho}$

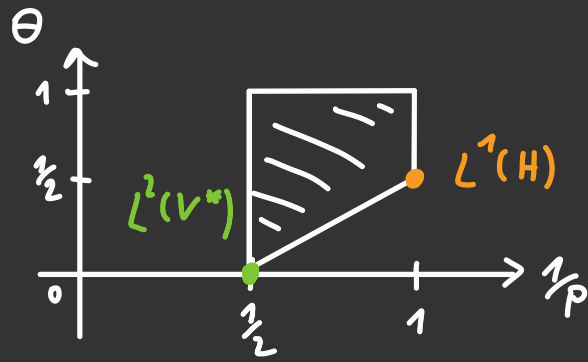
$L^1 \subseteq V^* \iff V \subseteq L^\infty$

Sobolev embedding
borderline case :-C

with $\rho = 2$, $\beta = 2/3$
fullfils $(*)$

What to do ?

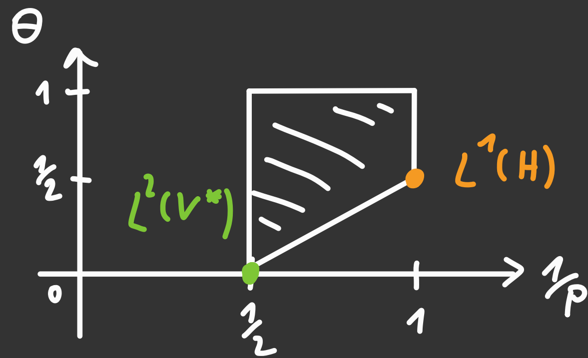
extended linear theory



Fix (p, θ) from arc

(say "admissible")

<u>extended linear theory</u>



Fix (p, θ) from area

(say "admissible")

<u>Claim:</u> $du - \Delta u \, dt = f \, dt + g \, dW(t)$

$$u(0) = u_0$$

well-posed for $f \in L^p([0,T]; X_\theta)$

$\rightsquigarrow$ need a-priori estimate

A strange coercivity condition

consider

$$du + \tilde{A}(u)\, dt = \tilde{B}(u)\, dW(t), \qquad u(0) = u_0 \qquad \text{(VP)}$$

satisfying for $\eta > 0$:

$$\langle \tilde{A}(u), u \rangle - \left(\tfrac{1}{2} + \eta\right) \| \tilde{B}(u) \|_{\mathcal{L}_2}^2$$

$$\geq \kappa \|u\|_V^2 - \phi(t) \|u\|_H^2 - \psi(t) - \sum_i \psi_i(t) \|u\|_{1-\theta_i},$$

integrable

where $\psi_i \in L^{p_i}([0,T])$ and (p_i, θ_i) admissible

A-priori estimate

Prop.: u sol. to (VP) satisfying coercivity condition (+ moment condition)

Then $\exists C = C(T, \eta, k, \|\phi\|_{L^1}) > 0$:

$$\mathbb{E} \sup_t \|u(t)\|_H^2 + \mathbb{E} \int_0^T \|u(t)\|_V^2 \, dt + \mathbb{E} \int_0^T \|\tilde{B}(u)\|_{\mathcal{L}_2}^2 \, dt$$

$$\leq C \Big(\mathbb{E} \|u_0\|_H^2 + \mathbb{E} \int_0^T \psi(t) \, dt + \sum_i \mathbb{E} \Big(\int_0^T \psi_i(t)^{p_i} \, dt \Big)^{2/p_i}$$

A-priori estimate

Prop.: u sol. to (VP) satisfying coercivity condition (+ moment condition)

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$$\leq C \Big(\mathbb{E} \|u_0\|_H^2 + \mathbb{E} \int_0^T \psi(t) \, dt + \sum_i \mathbb{E} \Big(\int_0^T \psi_i(t)^{p_i} \, dt \Big)^{2/p_i} \Big)$$

Rem.: produce terms $\varepsilon \, \mathbb{E} \sup_{s \in [0,t]} \|u(s)\|_H^2$, can only absorb at very end!

New Criticality Condition

$\exists\, s \geq 0,\ \beta \in (\tfrac{1}{2}, 1],\ \alpha \in [0, \tfrac{1}{2}]: \quad (1+s)(2\beta - 1) \leq 1 + 2\alpha$

and:

$$\|F(u)\|_\alpha \lesssim 1 + \|u\|_\beta^{1+s}, \qquad \|F(u) - F(v)\|_\alpha \lesssim (1 + \|u\|_\beta^s + \|v\|_\beta^s)\|u-v\|_\beta .$$

Back to weak Allen–Cahn in $n=2$: Can take <u>any</u> $\alpha \in [0, \tfrac{1}{2}]$.

E.g. $\alpha = \tfrac{1}{2},\ \beta = \tfrac{5}{6},\ s = 2$.

$$\|F(u)\|_\alpha = \|F(u)\|_{L^2} = \|u\|_{L^6}^3 \lesssim \|u\|_\beta^{1+s}$$

and $(1+2)(2 \cdot \tfrac{5}{6} - 1) = 2 = 1 + 2 \cdot \tfrac{1}{2}$ ✓ $\square$

Thank you for your attention!