Existence for quasilinear systems of SPDEs in a variational setting

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Our problem for today

Consider SPDE

$$\begin{aligned} du &= \left[\partial_i(a_{ij}(u)\partial_j u) + \partial_i \Phi_i(u) + \phi(u)\right] dt \\ &\quad + \sum_{n\geq 1} \left[b_{n,j}(u)\partial_j u + g_n(u)\right] dw_n, \\ u(0) &= u_0, \end{aligned}$$

on $D \subseteq \mathbb{R}^d$ subject to Dirichlet BC.

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$$du^\alpha = \left[\partial_i(a_{ij}^{\alpha\beta}(u)\partial_j u^\beta) + \partial_i \Phi_i^\alpha(u) + \phi^\alpha(u)\right]dt + \sum_{n\geq 1}\left[b_{n,j}^{\alpha\beta}(u^\beta)\partial_j u^\beta + g_n^\alpha(u)\right]dw_n,$$

$$u^\alpha(0) = u_0^\alpha,$$

on $D \subseteq \mathbb{R}^d$ subject to Dirichlet BC, $\alpha = 1, \dots, N$.

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Coefficients $a_{ij}^{\alpha\beta}$, $b_{n,j}^{\alpha\beta}$ are symmetric, no smoothness, and elliptic:

$$\left(a_{ij}^{\alpha\beta}(t,x,y) - \frac{1}{2} b_{n,i}^{\gamma\alpha}(t,x,y^\alpha) b_{n,j}^{\gamma\beta}(t,x,y^\alpha)\right)\xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2.$$

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Question

Does a solution to this system of SPDEs exist?

Some inspiration from the deterministic world

Disser, ter Elst, Rehberg JDE '17

$$\begin{aligned} u' - \partial_i(a_{ij}(u)\partial_j u) &= f, \\ u(0) &= 0, \end{aligned}$$

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Tools used:

1. well-posedness for linear equation with $f \in L^p(0, T; H^{-1}(D))$ where $p > 2$,
2. Schauder's fixed point theorem.

deterministic extrapolation result . . .

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. . . and why it fails for SPDEs!

deterministic extrapolation result ...

... and why it fails for SPDEs!

For $p \in (1, \infty)$ put

$$E_p = L^p(0, T; H^{-1}(D)),$$
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Parabolic operator

$$\partial_t - \partial_i(a_{ij}\partial_j) \colon V_p \to E_p$$

bounded for all $p \in (1, \infty)$.

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Moreover: $\partial_t - \partial_i(a_{ij}\partial_j)$ invertible $V_2 \to E_2$ by Lax–Milgram lemma.

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Fact: $(E_p)_{p\in(1,\infty)}$ and $(V_p)_{p\in(1,\infty)}$ are complex interpolation scales

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Lemma (Sneiberg)

Let T bounded between interpolation scales $(X_i)_{i\in(a,b)}$ and $(Y_i)_{i\in(a,b)}$. If $T\colon X_{i_*}\to Y_{i_*}$ invertible, then $T\colon X_i\to Y_i$ invertible for all $i\in(i_*-\varepsilon, i_*+\varepsilon)$.

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Upshot: $\partial_t - \partial_i(a_{ij}\partial_j)$ invertible $V_{2+\varepsilon}\to E_{2+\varepsilon}$.

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$$du = (Au + f)\, dt + (Bu + g)\, dW.$$

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But: Set $E_p = L^p(\Omega \times (0, T); H^{-1}(D)) \times L^p(\Omega \times (0, T); \mathcal{L}_2(U, L^2(D)))$, V_p analogous, solution operator

$$S\colon E_2 \ni (f, g) \mapsto u \in V_2$$

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Here we can attack! (later)

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Ordinary variational regularity: $V_2 \subseteq C([0,T]; L^2(D))$
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For $p > 2$: Sobolev + Arzela–Ascoli

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Mapping $v \mapsto u$ compact $\implies$ has fixed point u (by Schauder) such that

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One has: $V_p = L^p(\Omega; L^p(0,T; H^1(D)) \cap C([0,T]; B^{1-2/p}_{2,p}(D)))$.

No topology on probability space Ω $\rightsquigarrow$ compactness more delicate!

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4. Identify u as solution of original SPDE.

Approximation of Debussche–Hofmanova–Vovelle

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If only we could extrapolate variational regularity ...


YES, WE CAN!

extrapolation of variational solutions

Let

- $V \subseteq H \subseteq V^*$ Gelfand triple,
- W a U-cylindrical Brownian motion,
- $A \colon \Omega \times (0, T) \to \mathcal{L}(V, V^*)$ symmetric, bounded,
- $B \colon \Omega \times (0, T) \to \mathcal{L}_2(U, H)$ bounded.

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$$\begin{aligned} du &= (Au + f)\, dt + (Bu + g)\, dW, \\ u(0) &= u_0. \end{aligned}$$

Only assume ellipticity:

$$\langle Av, v \rangle - \frac{1}{2} \|Bv\|^2_{\mathcal{L}_2(U,H)} \geq \lambda \|v\|^2_V - M \|v\|^2_H.$$

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(Can reduce to $M = 0$.)

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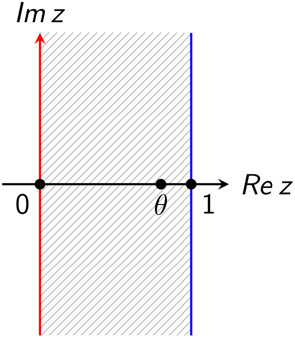
- $S \subseteq \mathbb{C}$ unit strip.
- Craft $S \ni z \mapsto (A(z), B(z))$ analytic
- with $A(\theta) = A$ and $B(\theta) = B$ for $\theta \in (0, 1)$.



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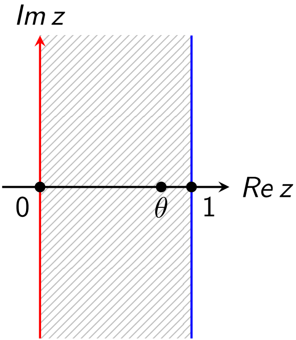

Red line: perturbation of autonomous case
$\rightsquigarrow$ L^p-maximal regularity for all p

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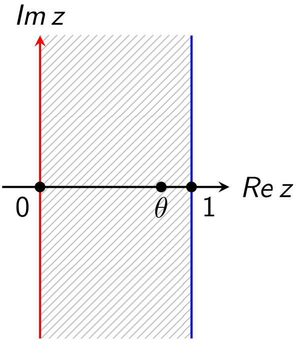

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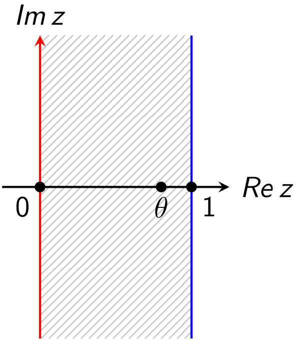

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Upshot: Stein interpolation of solution operator between L^2 and L^p
$\Longrightarrow$ $L^{2+\varepsilon}$-maximal regularity for $(A(\theta), B(\theta)) = (A, B)$.

extrapolation of variational solutions – main results

Theorem (B., Veraar)

Exists $p > 2$ depending on ellipticity of (A, B) such that for

$$f \in L^p(\Omega \times (0,T); V^*),\ g \in L^p(\Omega \times (0,T); \mathcal{L}_2(U,H)),$$
$$u_0 \in L^p(\Omega; (H,V)_{1-2/p,p})$$

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Theorem (B., Veraar)

The laws of

$$\left\{ u \text{ solution} \colon (A,B) \text{ uniformly elliptic}, \|f\|, \|g\|, \|u_0\| \le K \right\}$$

are tight on $C([0,T]; H)$.

What about this symmetry assumption?

Fix reference operator A_0 (for example $1 - \Delta$).

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Way out: symmetry of $A \implies$ no bad signs!

back to the start

Recall our system of SPDEs

$$
\begin{aligned}
du^\alpha &= \left[\partial_i(a_{ij}^{\alpha\beta}(u)\partial_j u^\beta) + \partial_i \Phi_i^\alpha(u) + \phi^\alpha(u)\right] dt \\
&\quad + \sum_{n\geq 1} \left[b_{n,j}^{\alpha\beta}(u^\beta)\partial_j u^\beta + g_n^\alpha(u)\right] dw_n, \\
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back to the start

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$$\begin{aligned} du^\alpha &= \left[\partial_i(a_{ij}^{\alpha\beta}(u)\partial_j u^\beta) + \partial_i \Phi_i^\alpha(u) + \phi^\alpha(u)\right] dt \\ &\quad + \sum_{n\geq 1} \left[b_{n,j}^{\alpha\beta}(u^\beta)\partial_j u^\beta + g_n^\alpha(u)\right] dw_n, \\ u^\alpha(0) &= u_0^\alpha. \end{aligned}$$

Approximate problems: just regularize the coefficients!

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Approximate problems: just regularize the coefficients!

Extrapolated variational regularity + stochastic compactness

$$a.s.\ u_n \to u \text{ in } C([0,T];L^2(D)) \quad \& \quad \nabla u_n \rightharpoonup \nabla u \text{ in } L^p(0,T;L^2(D)).$$

back to the start

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Approximate problems: just regularize the coefficients!
Extrapolated variational regularity + stochastic compactness

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$$

Latter fact in general not useful, but:

∇u_n bounded in $L^p(\Omega \times (0,T);L^2(D)) \implies$ Vitali's convergence theorem applicable

main results on quasilinear SPDEs

Consider system of SPDEs

$$\begin{aligned} du^\alpha &= \left[\partial_i(a_{ij}^{\alpha\beta}(u)\partial_j u^\beta) + \partial_i \Phi_i^\alpha(u) + \phi^\alpha(u)\right] dt \\ &\quad + \sum_{n\geq 1} \left[b_{n,j}^{\alpha\beta}(u^\beta)\partial_j u^\beta + g_n^\alpha(u)\right] dw_n, \\ u^\alpha(0) &= u_0^\alpha. \end{aligned}$$

Theorem (B., Veraar)

Let Φ, ϕ Lipschitz, $u_0 \in L^p(\Omega; B_{2,p,0}^{1-2/p}(D)) \implies$ system admits solution.

Theorem (B., Veraar)

Assume system diagonal, Φ, ϕ of polynomial growth, ϕ "dissipative" and $u_0 \in L^p(\Omega; B_{2,p,0}^{1-2/p}(D)) \cap L^q(\Omega \times D) \implies$ system admits solution.

Thank you for your attention!

A digital version of this presentation can be found here: