# Function spaces and interpolation theory

## under minimal geometric assumptions and with mixed boundary conditions

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Joint work with Moritz Egert (Orsay)

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## Problem

$$\left[H^{s_0,p_0}(O), H_D^{s_1,p_1}(O)\right]_\theta = ? \qquad \theta \in (0, 1).$$

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## Solution?

$$\left[H^{s_0,p_0}(O), H_D^{s_1,p_1}(O)\right]_\theta = H^{s,p}_{(D)}(O) \qquad \theta \in (0,1).$$

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3. "Gluing" interpolation scales

   Can glue together overlapping scales (Wolff's Theorem).

   **Example**: $\big\{H^{s,p}_{(D)}(O)\big\}_{\frac{1}{p}\neq s\in[0,1]}$ and $\big\{H^{s,p}_D(O)\big\}_{s\in(\frac{1}{p},1+\frac{1}{p})}$ overlap.

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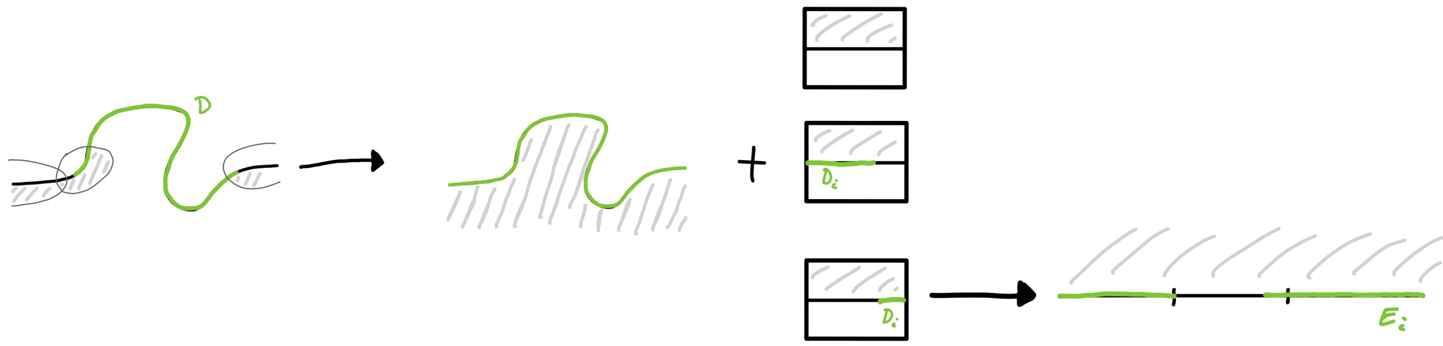
Used minimal geometric assumption in first step (later!).

# Hard inclusion – A first reduction: Localization

- Geometric:



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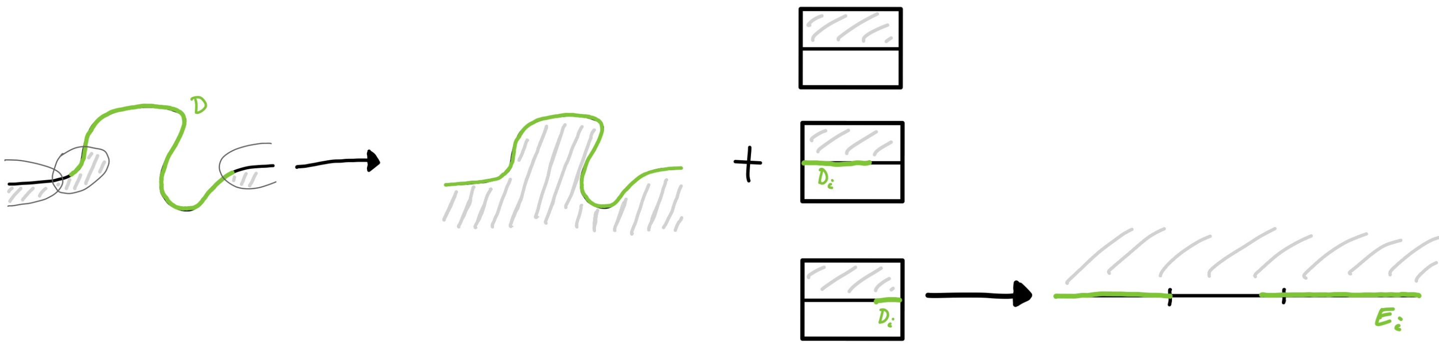


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$$H^{s,p}_{(D)}(O) \underset{R}{\overset{E}{\rightleftarrows}} \mathbb{H}^{s,p}_{(E)}(O) = \Big( \bigtimes_i H^{s,p}_{(E_i)}(\mathbb{R}^d_+) \Big) \times H^{s,p}_{(\partial O)}(O)$$

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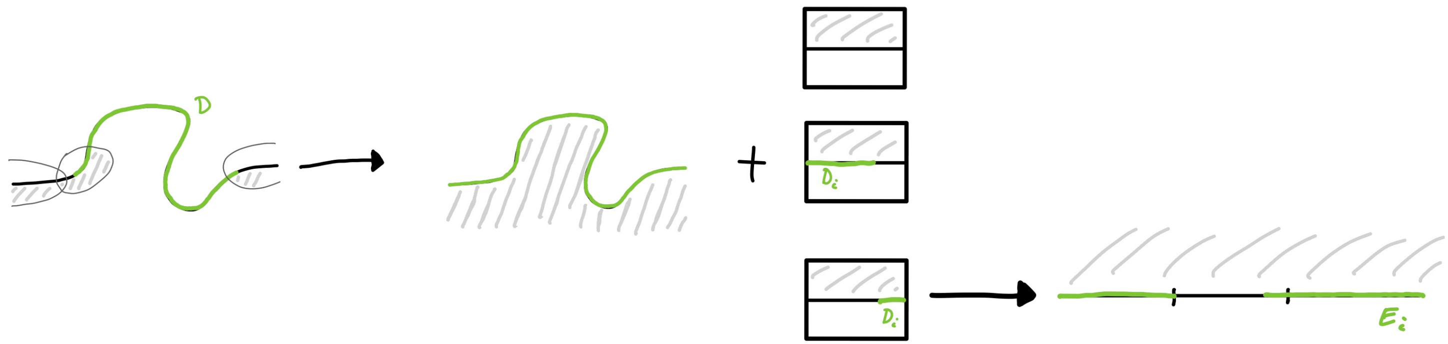


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$\mathbb{H}^{s,p}_{(E)}(O)$ interpolates component wise, so:

$$\left.\begin{array}{ll} & \text{pure Dirichlet on } O \\ \& & \text{mixed BC on } \mathbb{R}^d_+ \end{array}\right\} \implies \text{interpolation of } H^{s,p}_{(D)}(O).$$

# Bullet spaces: The key to both cases

Let $U \subseteq \mathbb{R}^d$ closed, put (with $X \in \{H, W\}$)

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3. Restriction to $O$ retraction: $O$ full dimensional.

# Mixed BC on $\mathbb{R}^d_+$: Another reduction

1. Step 1: $s$ small.

   Reduces to pure Dirichlet situation (even on O):

$$H^{s,p}(O) = \big[L^{p_0}, H^{1,p_1}_{\partial O}(O)\big]_s \subseteq \big[L^{p_0}, H^{1,p_1}_{D}(O)\big]_s.$$

2. Step 2: Work in the boundary $\mathbb{R}^{d-1}$.

   Write

$$f = \underbrace{f - ERf}_{\text{pure Dirichlet BC} \checkmark} + \underbrace{ERf.}_{\heartsuit \text{ of the matter}}$$

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   Which means: $\mathcal{H}^{d-1}(B(x,r) \cap E_i) \approx r^{d-1}$ for all $x \in E_i$ and $r \leq 1$.

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   Follows from porousity of $\partial D$ in $\partial O$.

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## Schema for the inclusion

$$\begin{array}{ccc}
H^{s,p}_{E_i}(\mathbb{R}^d_+) & & \Big[L^{p_0}(\mathbb{R}^d_+), H^{1,p_1}_{E_i}(\mathbb{R}^d_+)\Big]_s \\
\Big\downarrow \mathcal{R} & & \text{reiteration} \Big\| \\
 & & \Big[H^{\frac{1}{q}-\varepsilon,q}(\mathbb{R}^d_+), H^{1,p_1}_{E_i}(\mathbb{R}^d_+)\Big]_\eta \\
 & & \mathcal{E}\Big\uparrow \\
W^{s-\frac{1}{p},p}_\bullet({}^cE_i) & \overset{(\heartsuit)}{=} & \Big[W^{-\varepsilon,q}_\bullet({}^cE_i), W^{1-\frac{1}{p_1},p_1}_\bullet({}^cE_i)\Big]_\eta
\end{array}$$

# Summary

## Theorem (B., Egert, JFAA 2019)

*Let*

1. $O \subseteq \mathbb{R}^d$ *open,*
2. $O$ *and* ${}^c O$ *is* $d$*-regular,*
3. $D \subseteq \partial O$ *is* $(d-1)$*-regular,*
4. $\overline{\partial O \setminus D}$ *has uniform bi-Lipschitz charts, and*
5. $\partial D$ *is porous in* $\partial O$*.*

*Fix* $p_0, p_1 \in (1, \infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, $\theta \in (0,1)$ *and put*

$$s := (1-\theta)s_0 + \theta s_1 \quad \text{and} \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

*Then*

$$\left[H^{s_0,p_0}(O), H^{s_1,p_1}_D(O)\right]_\theta = H^{s,p}_{(D)}(O).$$

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Recall “easy inclusion”: Forgot BC and used Rychkov extension (need full dimensional $O$).

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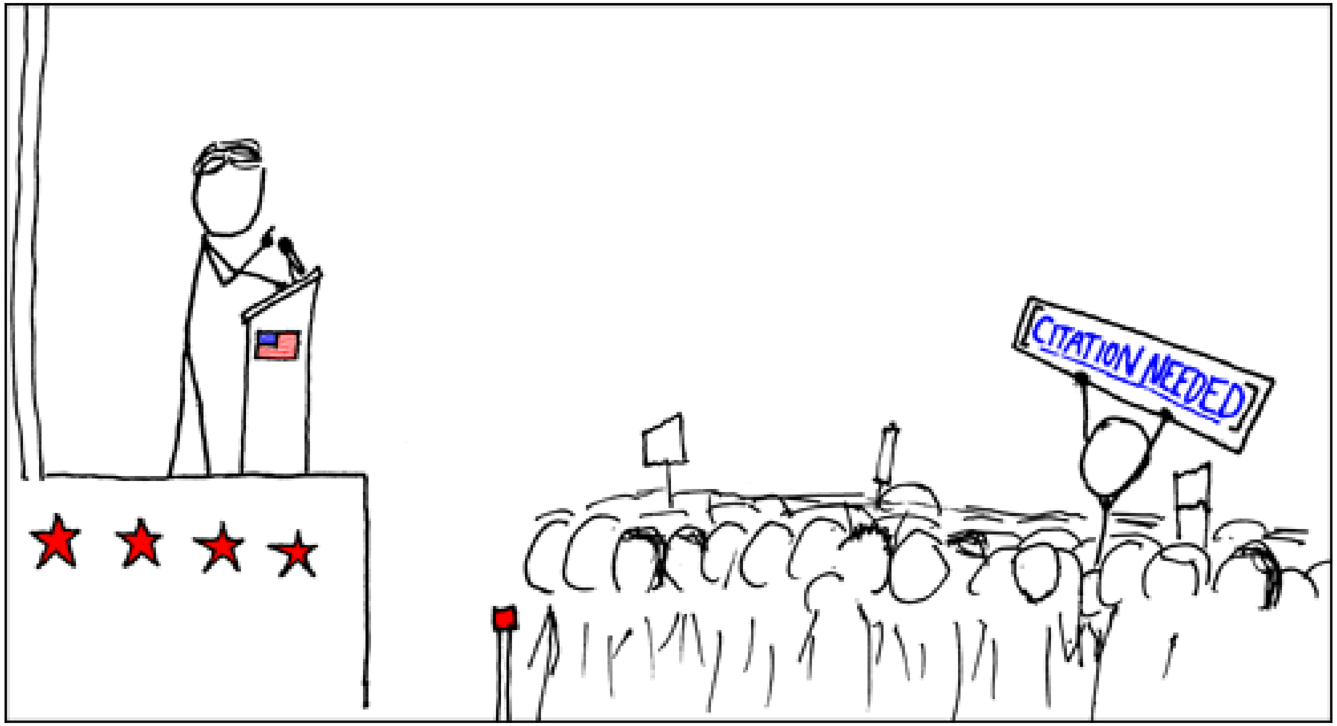
Can we benefit from BC?

# Thank you for your attention!




S. Bechtel and M. Egert.

*Interpolation theory for Sobolev functions*
*with partially vanishing trace on irregular open sets*.

J. Fourier Anal. Appl. (2019).