On mixed boundary conditions, function spaces, and Kato's square root property

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TECHNISCHE UNIVERSITÄT DARMSTADT

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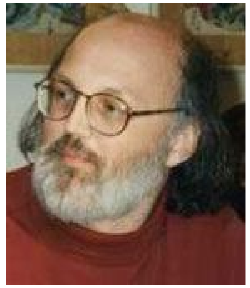
Interpolation theory

real & complex



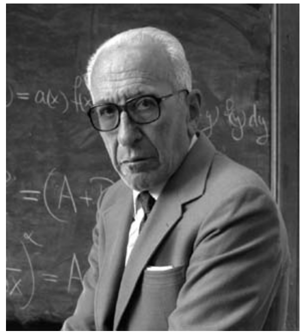
Extension operators

$W_D^{s,p}(O)$ & $W_D^{k,p}(O)$

Kato square root problem

Beyond CZ

$p > 2$ & $p < 2$



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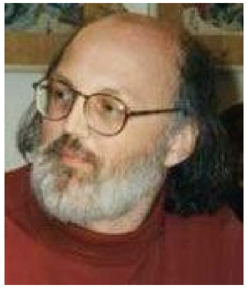

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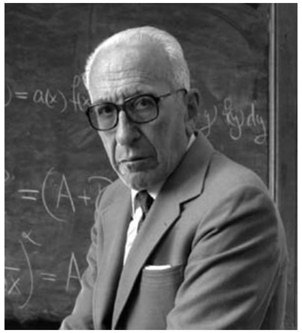

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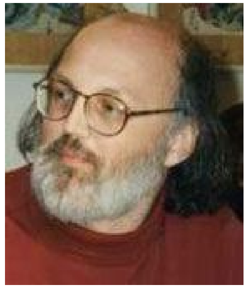

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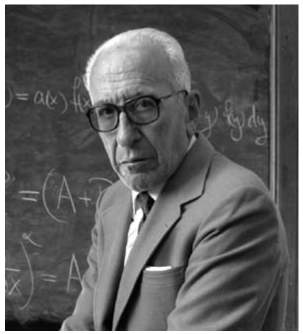

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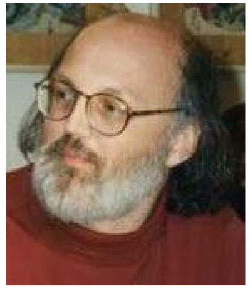

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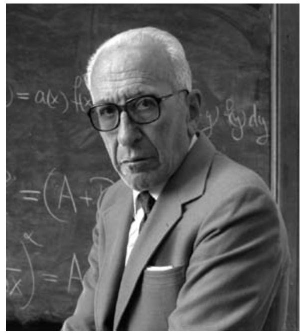

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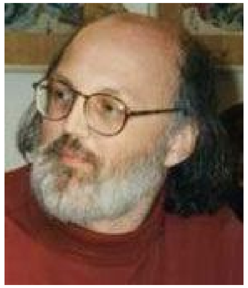

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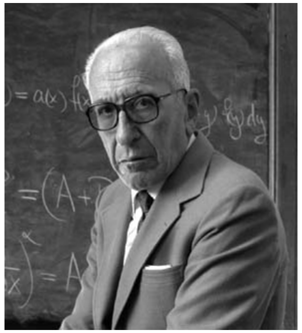

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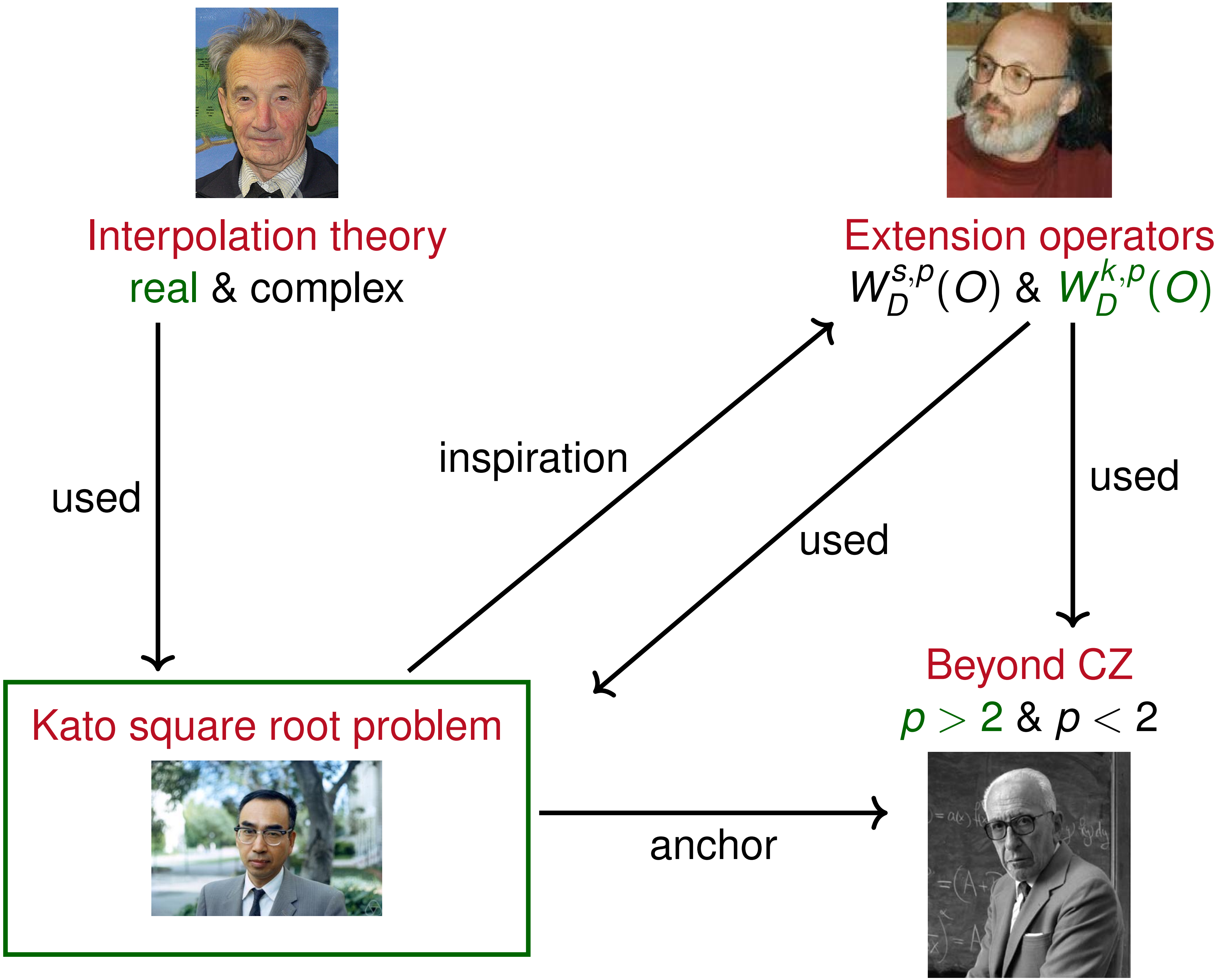
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- $a_{ij}, b_i, c_j, d : O \to \mathbb{C}^{m\times m}$ bounded and measurable
- define sesquilinear form on $\mathcal{V} \times \mathcal{V}$

$$a(u, v) = \int_O \sum_{i,j=1}^{d} a_{ij}\partial_j u \cdot \overline{\partial_i v} + \sum_{i=1}^{d} b_i u \cdot \overline{\partial_i v} + \sum_{j=1}^{d} c_j \partial_j u \cdot \overline{v} + du \cdot \overline{v}\, dx$$

- form a coercive in Gårding's sense

$$Re\, a(u, u) \geq \lambda(\|u\|^2_{\mathrm{L}^2(O)} + \|\nabla u\|^2_{\mathrm{L}^2(O)})$$

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Problem

For which spaces $\mathcal{V}$ do we have $D(L^{\frac{1}{2}}) = \mathcal{V}$ with equivalent norms?

What is known for mixed boundary conditions?

Theorem (AKM '06, EHT '16)

Suppose:

- O bounded domain
- O is d-regular
- ∂O is $(d-1)$-regular
- $D \subseteq \partial O$ is $(d-1)$-regular
- O is bi-Lipschitz near $\partial O \setminus D$

Then Kato's square root property holds for $\mathcal{V} = W_D^{1,2}(O)$.

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First-order approach

Put $\quad \Gamma := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \nabla_D & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & c \\ 0 & b & A \end{bmatrix}, \quad \Pi_B := \Gamma + \Gamma^* B.$

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$\Longrightarrow$ Π_B is bisectorial and $\Pi_B^2 = \begin{bmatrix} L & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$.

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For $U = \begin{bmatrix} v & 0 & 0 \end{bmatrix}^T$:

$$\|v\|_{W^{1,2}} \approx \|\Gamma U\| = \|\Pi_B U\| \approx \|\sqrt{\Pi_B^2} U\| = \|\sqrt{L} v\|.$$

The Axelsson–Keith–McIntosh framework

Provide: Sufficient conditions for square function estimate

$$\int_0^\infty \|t\Pi_B(1+t^2\Pi_B^2)^{-1}U\|^2 \frac{dt}{t} \approx \|U\|^2 \qquad U \in R(\Pi_B).$$

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square function estimate $\Longleftrightarrow$ bounded H^∞-calculus.

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Refinement by Egert–Haller-Dintelmann–Tolksdorf: square function estimate in mixed BC context

$$\begin{array}{ccc} & \text{EHT assumptions} & \\ \swarrow & & \searrow \\ \text{framework} & & \text{geometric} \end{array}$$

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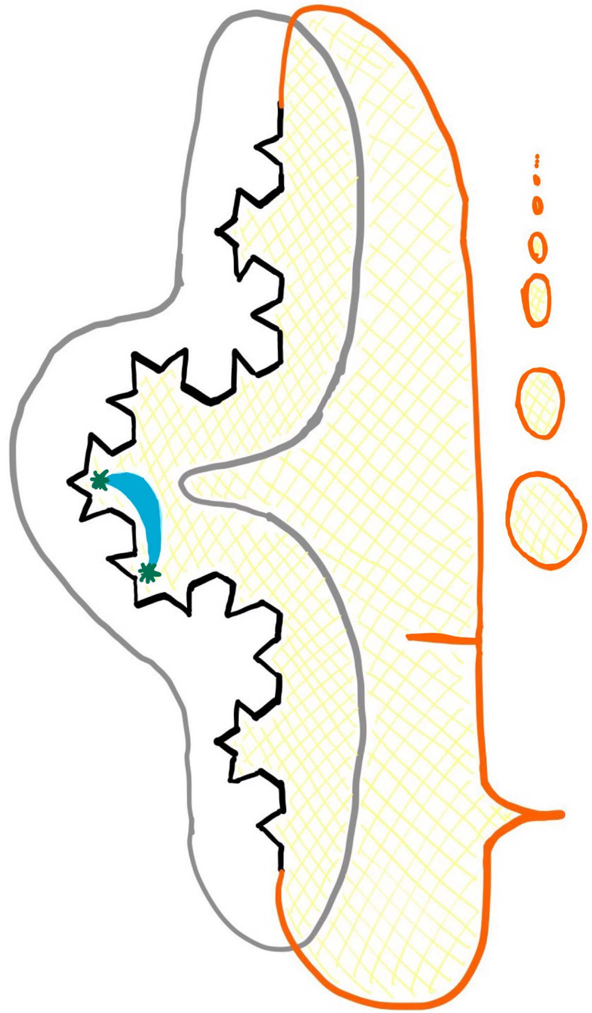

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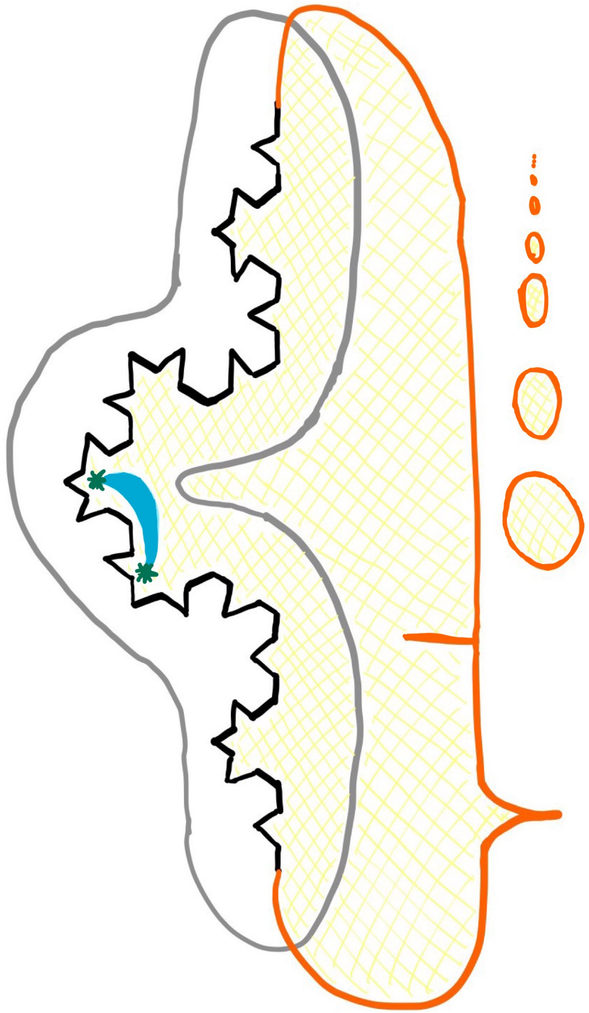

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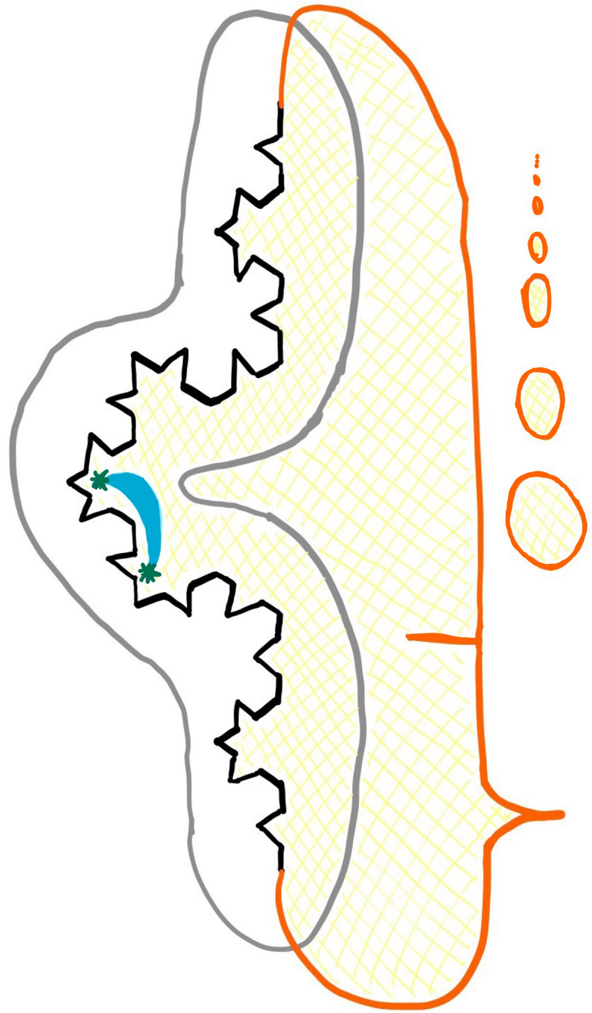

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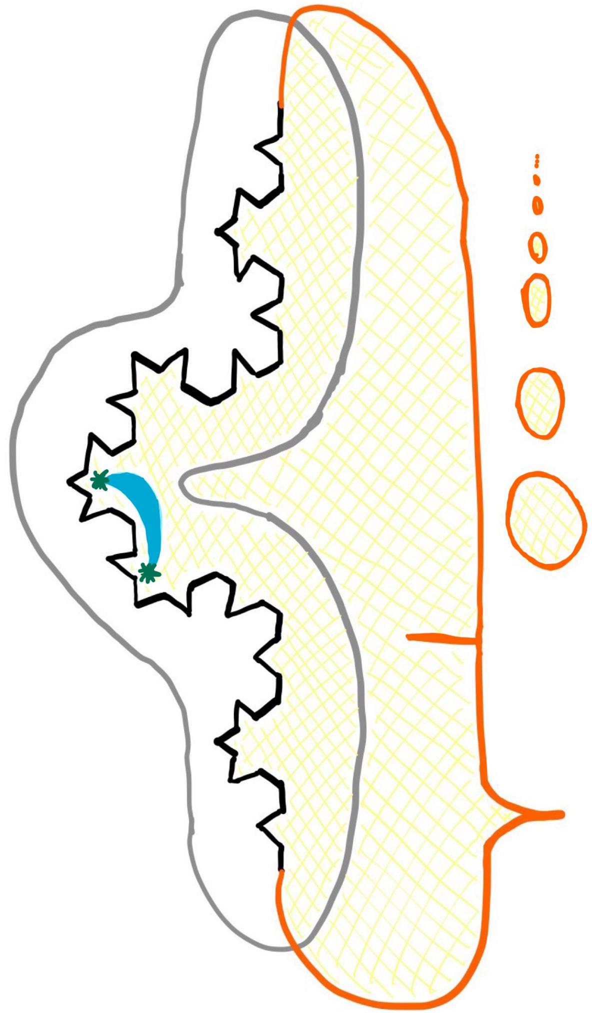

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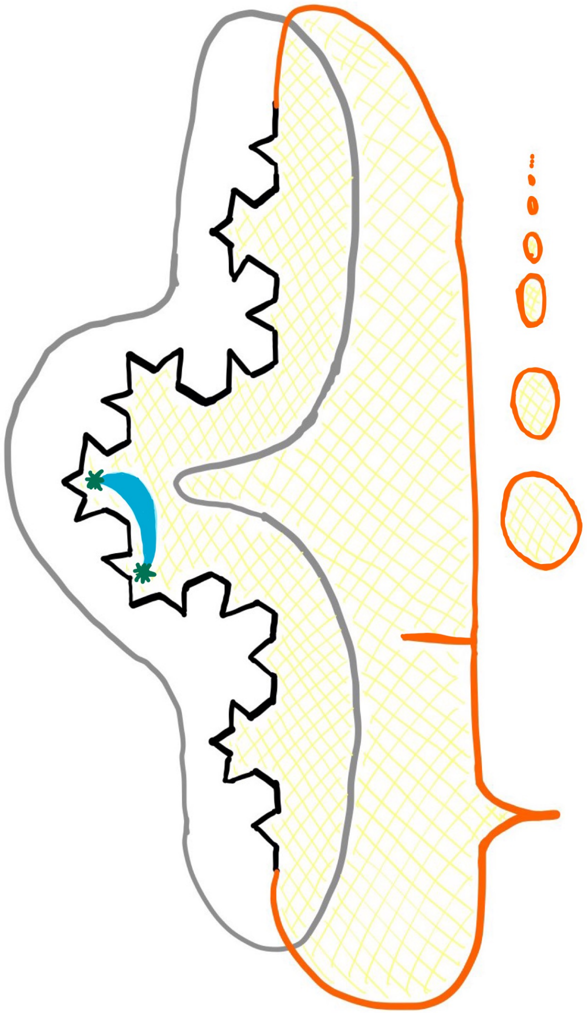

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Put $\Pi = \Gamma + \Gamma^*$. Whole space: Π accretive on its range:

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A glimpse on assumption (H7)

Put $\Pi = \Gamma + \Gamma^*$. Whole space: Π accretive on its range. Modify to:

$$\|(\Pi^2)^{\gamma/2} U\|_2 \gtrsim \|U\|_{[L^2, W^{1,2}]_\gamma} \qquad U \in R(\Gamma) \cap D(\Pi^2).$$

To check: Let

$$U \in R(\Gamma) \implies U = \begin{bmatrix} 0 \\ v \\ \nabla_D v \end{bmatrix} = \Pi V \quad \text{for} \quad V = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}.$$

Calculate:

$$\|U\|_{[L^2, W_D^{1,2}(O)]_\gamma} \overset{!}{\approx} \|U\|_{W^{\gamma,2}(O)} \lesssim \|v\|_{W_D^{1+\gamma,2}(O)} \overset{!}{\approx} \|(-\Delta_D + 1)^{1/2+\gamma/2} v\|$$
$$\approx \|(-\Delta_D + 1)^{\gamma/2} \Pi V\| = \|(\Pi^2)^{\gamma/2} U\|.$$

Real interpolation of Sobolev spaces

Theorem (B.–Egert JFAA '19)

- O open & d-regular with porous boundary,
- $D \subseteq \partial O$ uniformly $(d-1)$-regular,
- $p \in (1, \infty)$ and $s \in (0, 1) \setminus \{1/p\}$.

Then one has

$$(L^p(O), W^{1,p}_D(O))_{s,p} = \begin{cases} W^{s,p}_D(O) & (\text{if } s > 1/p) \\ W^{s,p}(O) & (\text{if } s < 1/p) \end{cases}.$$

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- Based on Grisvard's trace method.

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Impressions from Orsay 2018

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week 1

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Optimal elliptic regularity using extrapolation

Want to show: $D((1-\Delta_D)^{1/2+\gamma/2}) = W_D^{1+\gamma,2}(O)$.

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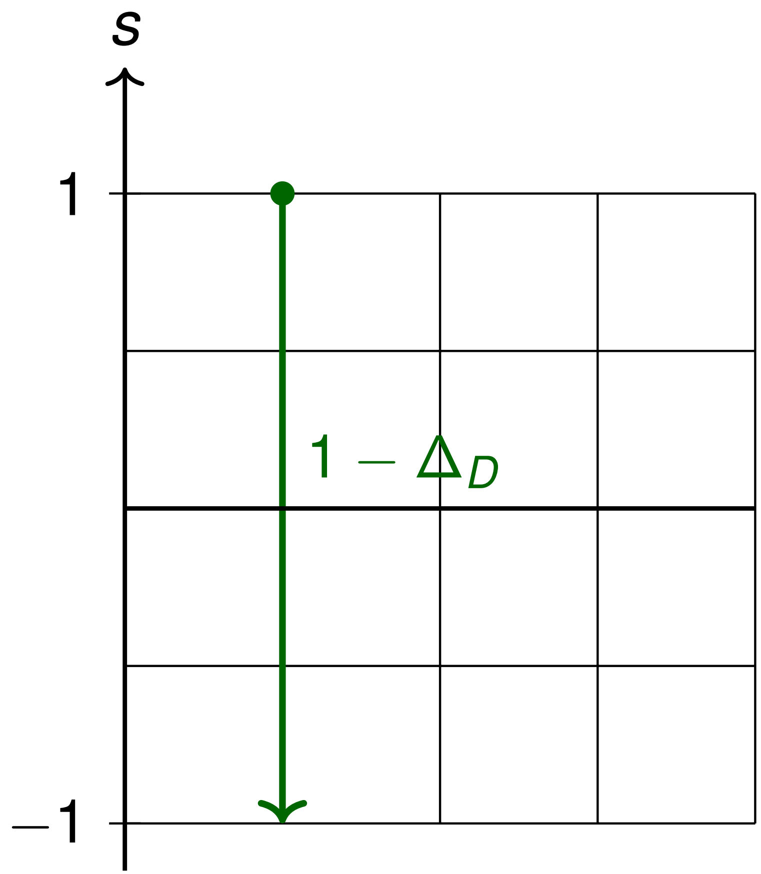

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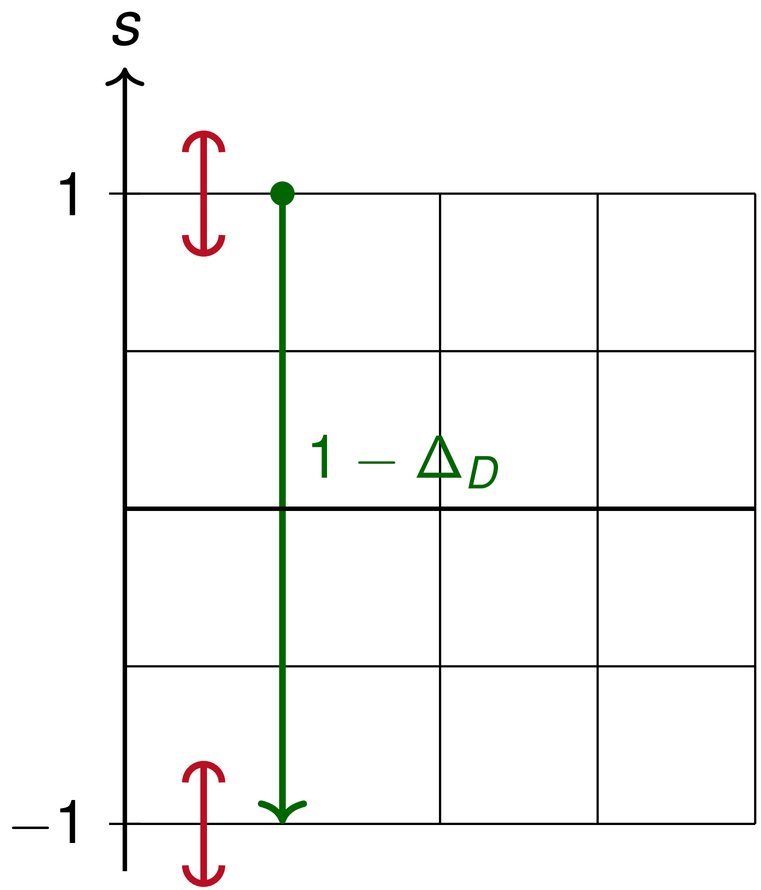

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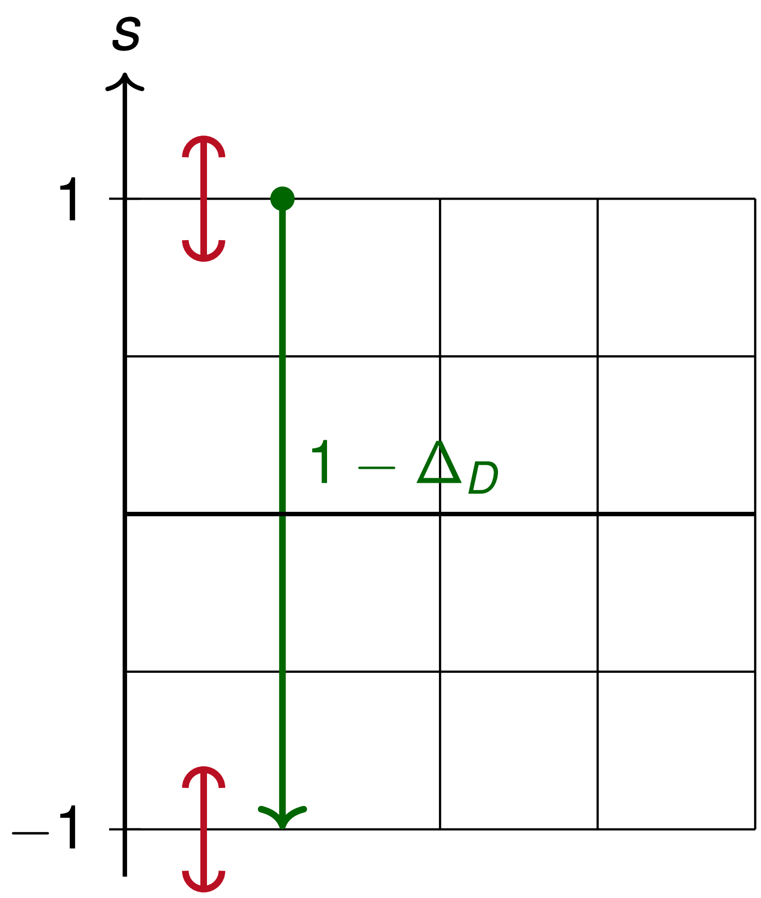

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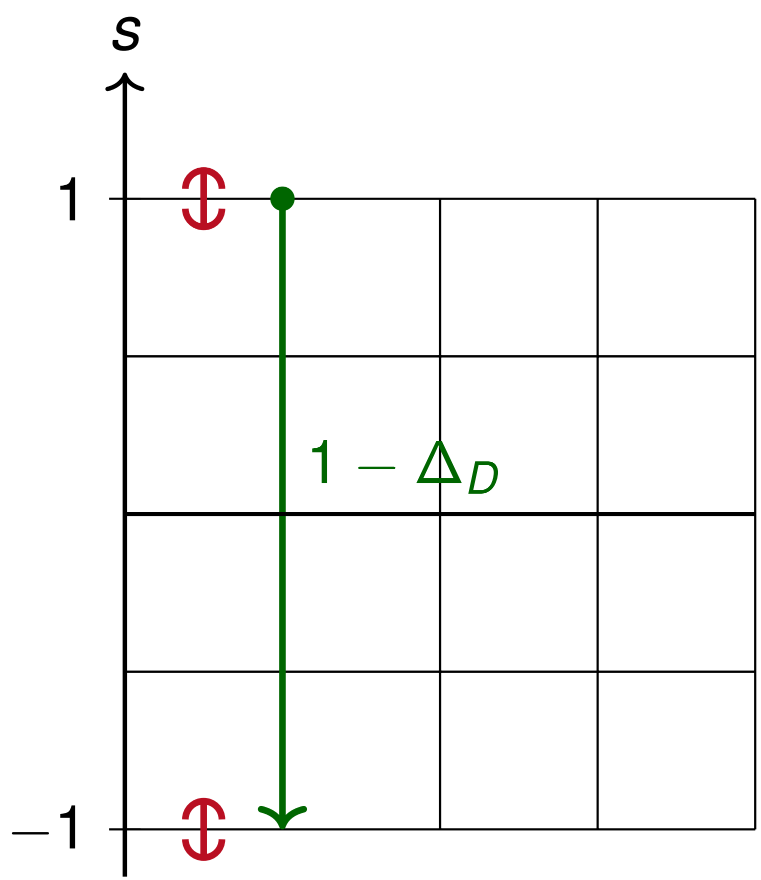

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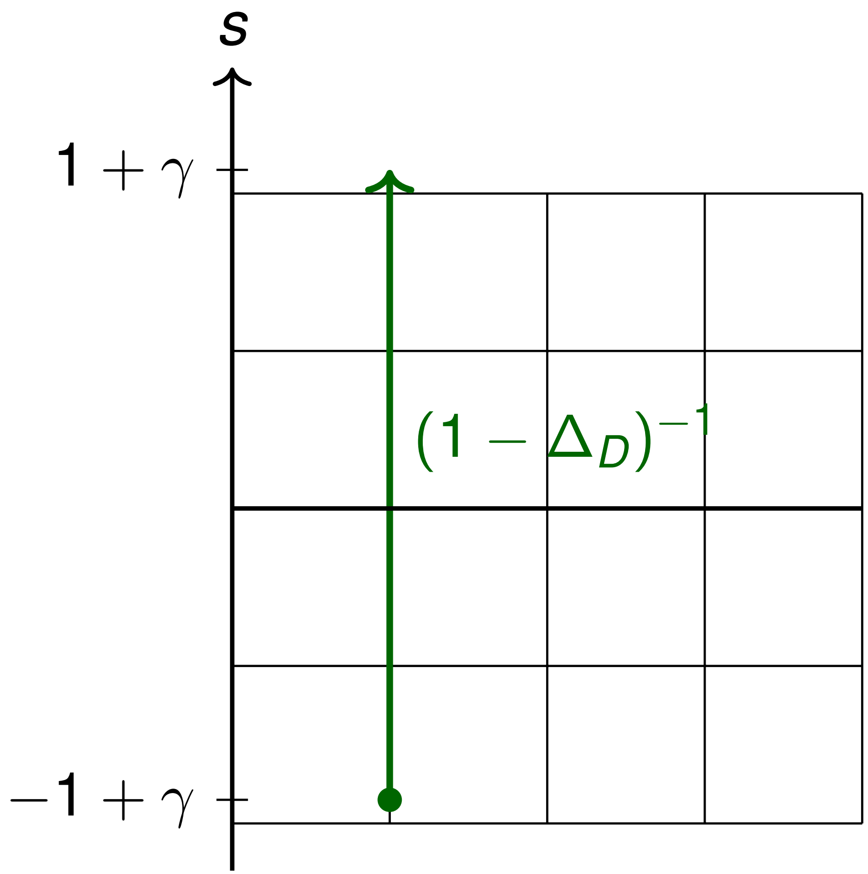

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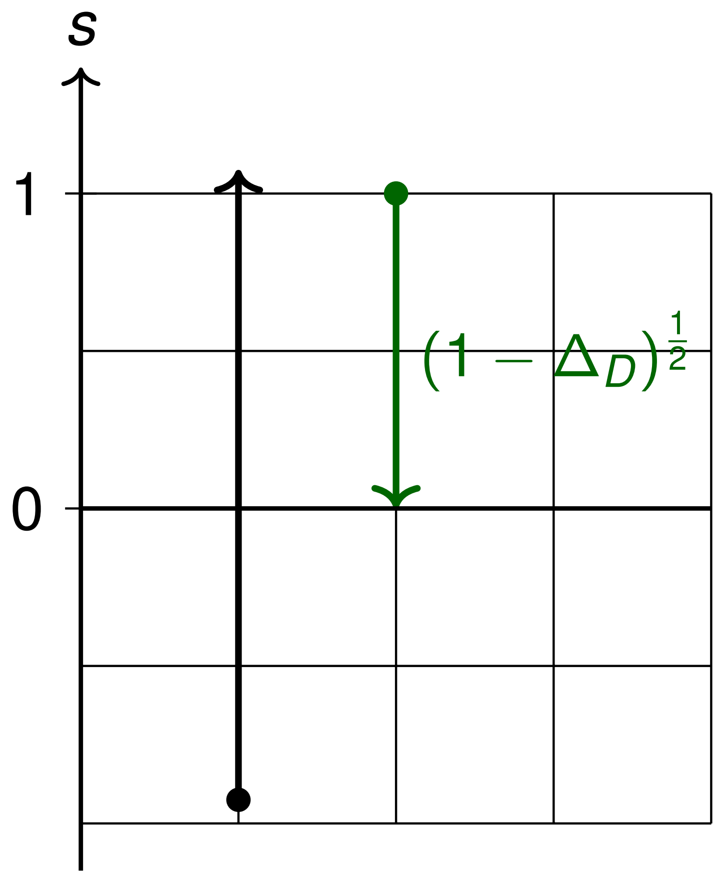

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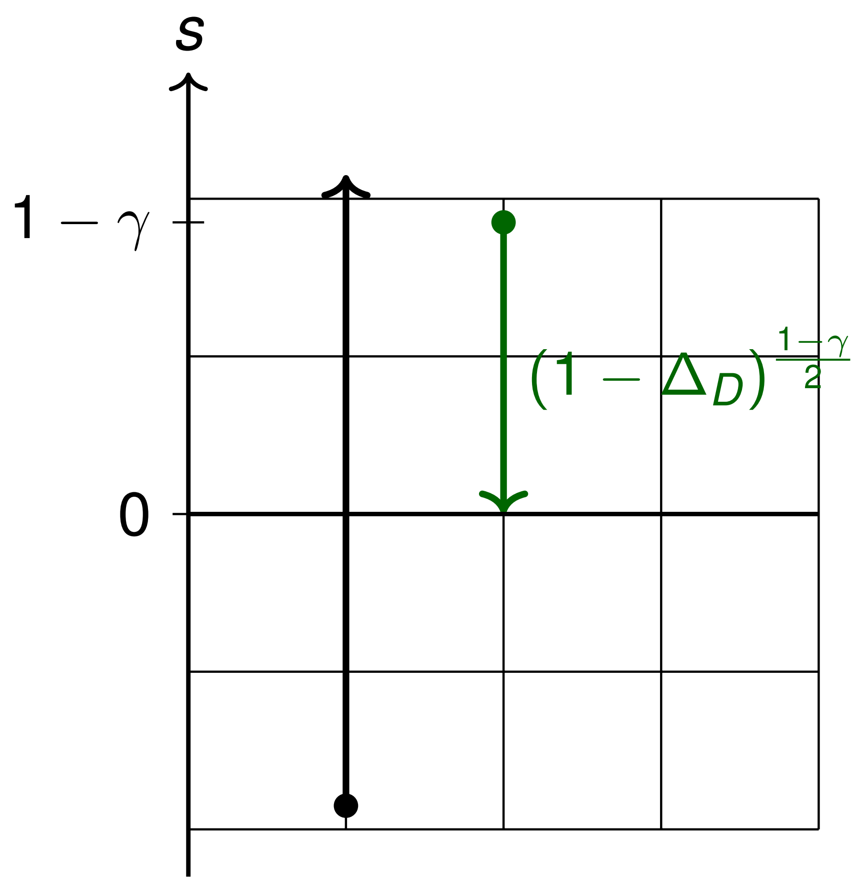

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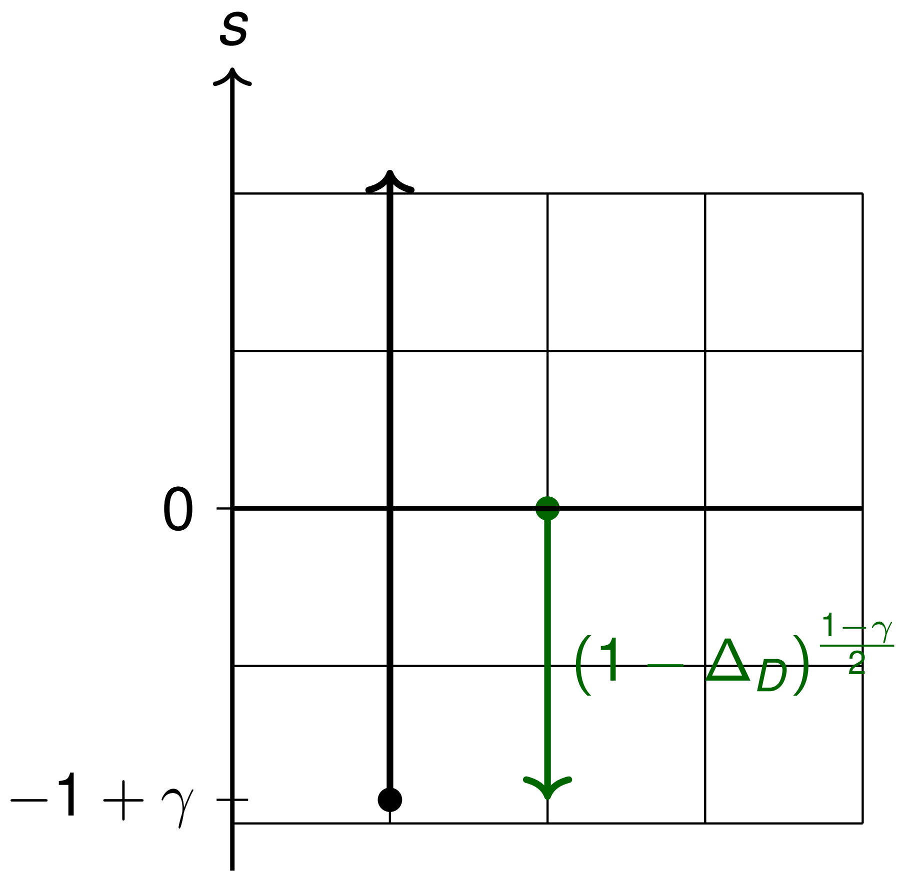

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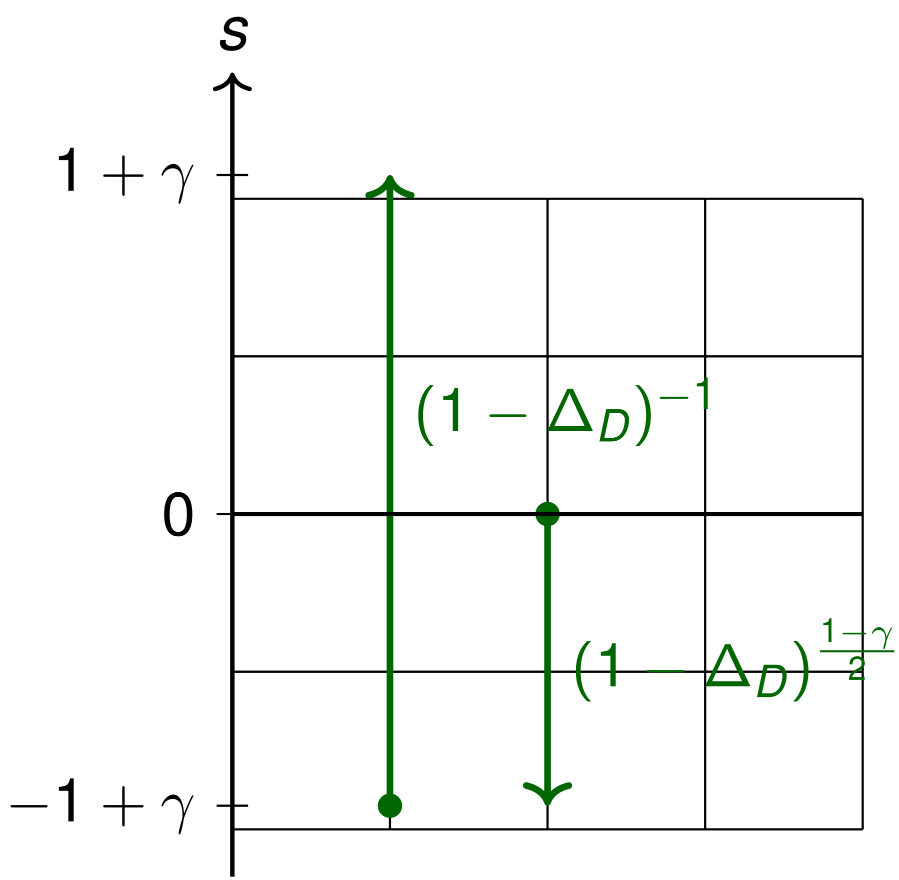

What is known for mixed boundary conditions?

Theorem (AKM '06, EHT '16)

Suppose:

- O ~~bounded domain~~
- O is d-regular

“domain”

- ∂O is ~~$(d-1)$-regular~~ porous
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boundary

Then Kato's square root property holds for $\mathcal{V} = W^{1,2}_D(O)$.

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Upshot: Kato on interior thick $\boldsymbol{O}$ gives Kato on (possibly thin) O_i.

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Thank you for your attention!