weighted non-autonomous maximal $L^q(L^p)$-regularity

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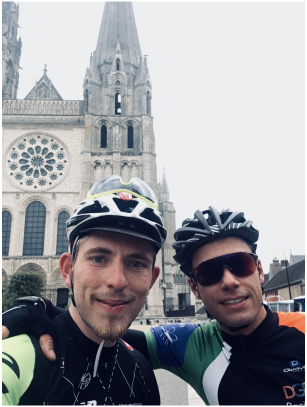
TUDelft

Where it all started. . .



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$$\partial_t u(t,x) - div_x A(t,x)\nabla_x u(t,x) = f(t,x), \qquad (t,x) \in (0,T) \times \mathbb{R}^d,$$
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Integrate:

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Representation formula: $u(t) = S_1(u)(t) + S_2(f)(t)$.

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$$(\tau, s) \mapsto L_s (2\pi i\tau + L_s)^{-1}.$$

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Actual bounds for symbol: good elliptic theory (later!)

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$$\|L_t S_1(u)(t)\|_2 \lesssim \int_0^t (t-s)^{-1+\varepsilon}\|u(s)\|_{H^1} ds \quad \implies \quad \text{Young}$$

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For convolution: $-1 \overset{!}{=} \beta + \frac{-3+\alpha}{2}$ iff. $2\beta + \alpha = 1$ (parabolic relation).

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$L^q(H^{-1,p})$-situation

Most important idea in this project!

Serious technical difficulty: to many derivatives on u!

Idea: Smoothing of the coefficients, then take the limit.

$L^2(H^{-1})$-situation

Lions: implicit constants uniform in ellipticity ✓

$L^q(H^{-1,p})$-situation

Perturbation methods $\Longrightarrow$ weak solutions, but not uniform in coefficient functions ☹

Dong–Kim framework

Theorem (Dong & Kim)

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Note: This uses the structure of the problem!

Main result

Theorem (B. & F. Gabel)

Let $p, q \in (1, \infty)$

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$$A \in \begin{cases} C_t^{\beta+\varepsilon}(H_x^{\alpha+\varepsilon, \frac{d}{\alpha}}), & \text{if } p < \frac{d}{\alpha}, \\ C_t^{\beta+\varepsilon}(C_x^{\alpha+\varepsilon}), & \text{else.} \end{cases}$$

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Given $f \in L^q(w; L^p)$, *there exists a unique solution to*

$$\partial_t u(t,x) - div_x A(t,x) \nabla_x u(t,x) = f(t,x), \qquad (t,x) \in (0,T) \times \mathbb{R}^d,$$
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with $\|L_t u(t)\|_{L^q(w;L^p)} \lesssim \|f\|_{L^q(w;L^p)}$.

Thanks for your attention!

A digital version of this presentation can be found here: