What we know about square roots of elliptic systems... and a bit more!

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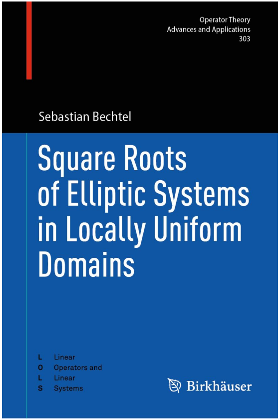

joint work (in part) with

R. Brown (Lexington), M. Egert, R. Haller (Darmstadt), P. Tolksdorf (KIT)

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Fix d (dimension) and m (system size). Let $O \subseteq \mathbb{R}^d$. Consider (formally):

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$$\mathfrak{a} : V \times V \to \mathbb{C}, \qquad a(u, v) = \int_O \begin{bmatrix} d & c \\ b & A \end{bmatrix} \begin{bmatrix} u \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \nabla v \end{bmatrix}} \, dx$$

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Examples for V: $H_0^1(O)$ (Dirichlet BC), $H^1(O)$ (natural BC),
$H_D^1(O)$ (mixed BC).

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Associate operator L_0 in $L^2(O)$ with $\mathfrak{a}$ via

$$u \in D(L_0) \;\&\; L_0 u = w \in L^2(O) \quad \Leftrightarrow \quad (w \mid v) = a(u,v) \quad \text{for all } v \in V.$$

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L is sectorial of angle $\omega \in [0, \pi)$: still nice, but maybe no generator of semigroup.

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Then:

- $u \in D(L)$ (product rule, coefficients cancel out)
- $u' = A^{-1} = 1 - \frac{1}{2} 1_{[0,1]}$ on $[0, 2] \implies u' \notin H^1(\mathbb{R})$, no optimal elliptic regularity!

Why are rough coefficients natural?

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2. Elliptic BVP over Lipschitz graph

What is Kato's square root problem?

L sectorial operator $\implies$ exists operator $L^{\frac{1}{2}}$ such that $L = L^{\frac{1}{2}} L^{\frac{1}{2}}$ (by functional calculus). Call $L^{\frac{1}{2}}$ the square root of L.

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Conjecture

One has

$$D(L^{\frac{1}{2}}) = V \quad \& \quad \|L^{\frac{1}{2}} u\|_{L^2(O)} \approx \|u\|_V.$$

Implicit constants depend on ellipticity, dimensions and V only.

Geometrical framework

Let $O \subseteq \mathbb{R}^d$ open, $D \subseteq \partial O$, $N = \partial O \setminus D$.

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2. D is Ahlfors–David regular.

In particular: no thickness, no boundedness, no connectedness required.

Solution to Kato's square root problem

Theorem (B., Egert, Haller-Dintelmann, Adv. Math.)

Let $O \subseteq \mathbb{R}^d$ open, $D \subseteq \partial O$, $N = \partial O \setminus D$. Assume D is Ahlfors–David regular and O is locally uniform near N. Then one has

$$D(L^{\frac{1}{2}}) = H^1_D(O) \quad \& \quad \|L^{\frac{1}{2}} u\|_{L^2(O)} \approx \|u\|_{H^1(O)}.$$

Implicit constants depend on ellipticity, dimensions and geometry only.

What about L^p-theory?

Reformulation of square root property:

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critical numbers

Define

$$I(L) = \{p \in (1,\infty) \colon \{(1+t^2L)^{-1}\}_{t>0} \text{ is } L^p\text{-bounded}\},$$
$$J(L) = \{p \in (1,\infty) \colon \mathcal{L}_0 \text{ is } p\text{-isomorphism}\}.$$

Put

$$p_-(L) = \inf I(L), \qquad p_+(L) = \sup I(L),$$
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- $O = \mathbb{R}^d$, Hölder coefficients $\implies \tilde{q}_+(L) = \infty$.
- In general: $\tilde{q}_+(L) \le 2 + \varepsilon$.

L^p-theory for square roots

Theorem

Assume setting of L^2-result.
Let $p \in (p_-(L), \tilde{q}_+(L))$. Then $L^{\frac{1}{2}} : W^{1,p}_D(O) \to L^p(O)$ isomorphism.

The interval $(p_-(L), \tilde{q}_+(L))$ is (essentially) optimal.

Consequences of square root estimates

1. Trace estimate for non-autonomous problems

   For $f \in L^2([0,T]; L^2(O))$ consider

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Important for applications to quasilinear problems with Neumann BC! (current project with T. Leeuwis and M. Veraar)

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   Condition $p > d$ fulfilled for:

A real, $m = 1$	general complex A
any dimension d	$d \leq 4$

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   Let $L_A = -div(A\nabla)$, $L_B = -div(B\nabla)$ (say with real coefficients). Does one have

$$\|L_A^{\frac{1}{2}} u - L_B^{\frac{1}{2}} u\|_{L^2(O)} \lesssim \|A - B\|_\infty \|u\|_{H^1_D(O)} \text{ ?}$$

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   But: Need complex structure (for analyticity) even for real statement!

Open questions / Ongoing projects

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Open questions / Ongoing projects

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   Current project with M. Egert and B. Kosmela (Darmstadt).

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   Student project with C. Hutcheson, T. Schmatzler, T. Tasci, and M. Wittig: recover L^2-main result with pure second-order proof.

Thank you for your attention!

A digital version of this presentation can be found here: