

A new framework to treat rough non-linear PDEs

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Non-linear model problem

Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}^n$, $|\Phi(u)| \lesssim |u|^{1+\rho}$, $|\Phi(u) - \Phi(v)| \lesssim (|u|^\rho + |v|^\rho)|u - v|$.



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Scaling argument suggests:

$$u_0 \in \dot{B}_{p,p}^\alpha \quad \text{with} \quad \frac{2}{\rho} = \frac{n}{p} - \alpha.$$



Weak solutions to (RD)

Definition

Call (u, T) a **weak solution** of

$$\partial_t u - \operatorname{div}(A \nabla u) = \Phi(u), \quad u(0) = u_0,$$

if

- $u, \nabla u \in L^2_{loc}((0, T) \times \mathbb{R}^n)$, $\Phi(u) \in L^{2^*}_{loc}((0, T) \times \mathbb{R}^n)$
- for all $\phi \in C_c^\infty((0, T) \times \mathbb{R}^n)$:

$$\iint -u \partial_t \phi + A \nabla u \cdot \nabla \phi = \iint \Phi(u) \phi$$



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- $u(t) \rightarrow u_0$ in the sense of distributions as $t \rightarrow 0$



Besov spaces and Z -spaces

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Let $p, q, r \in (0, \infty]$, $\beta \in \mathbb{R}$.

$$f \in Z_\beta^{p,q,r} \Leftrightarrow \left(\int_{t/2}^t \int_{B(x, \sqrt{t})} |s^{-\beta} f(s, y)|^r dy ds \right)^{\frac{1}{r}} \in L_*^q((0, \infty); L^p(\mathbb{R}^n)).$$



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Besov and Z -spaces (continued)

Still $u_0 \in \dot{B}_{p,q}^\alpha$. Put $\sigma = n/p - \alpha$. Then:

$$\mathcal{E}_L(u_0) \in Z_{\frac{\alpha}{2}}^{p,q,r} = Z_{\frac{n}{2p} - \frac{\sigma}{2}}^{p,q,r}, \quad r \in (0, \infty].$$



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Embeddings:

- $Z_{\frac{n}{2p} + \beta}^{p,q_0,r_0} \subseteq Z_{\frac{n}{2p} + \beta}^{p,q_1,r_1}$, if $q_0 \leq q_1$ and $r_0 \geq r_1$



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- $Z_{\frac{n}{2p_0} + \beta}^{p_0,q,r} \subseteq Z_{\frac{n}{2p_1} + \beta}^{p_1,q,r}$, if $p_0 \leq p_1$ (“Sobolev embedding”).



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- $Z_{\frac{n}{2p_0} + \beta}^{p_0,q,r} \subseteq Z_{\frac{n}{2p_1} + \beta}^{p_1,q,r}$, if $p_0 \leq p_1$ (“Sobolev embedding”).

Consequence: $u_0 \in \dot{B}_{p,p}^\alpha \implies \mathcal{E}_L(u_0) \in Z_{\frac{n}{2r} - \frac{\sigma}{2}}^{r,q}$, $r \geq p$, $q \in (1, \infty]$.



Wellposedness of weak solutions to (RD)

Set $RD_-(n, \rho) = \frac{\rho}{2}(n+2)$, $RD_+(n, \rho) = n(1+\rho)\frac{\rho}{2}$.

Theorem (Auscher–B.)

Let $\rho > 2/n$, $p \in (1, RD_+(n, \rho))$, $\alpha \in (-1, 0)$ satisfying $2/\rho = n/p - \alpha$ and some mild extra assumption. Fix $u_0 \in \dot{B}_{p,p}^\alpha$.

Then, there exists a weak solution u to (RD) which is unique and maximal with

$$u \in Z_{\frac{n}{2r} - \frac{1}{\rho}}^{r,q}, \quad \nabla u \in Z_{\frac{n}{2p} - \frac{1}{\rho} - \frac{1}{2}}^{p,2},$$

where $RD_-(n, \rho) < r < RD_+(n, \rho)$, $r \geq p$, $q > RD_-(n, \rho)$.



Mild solutions & existence

Define formally

$$\mathcal{L}_1^L(f)(t) = \int_0^t e^{-(t-s)L} f(s) ds.$$

Definition

Call u a **mild solution** if $u = \mathcal{E}_L(u_0) + \mathcal{L}_1^L(\Phi(u))$.



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$$\text{Let } u \in L^r_{\frac{n}{2r} - \frac{1}{\rho}} =: X \quad \implies \quad \Phi(u) \in L^{\frac{r}{1+\rho}}_{\frac{n(1+\rho)}{2r} - \frac{1}{\rho} - \frac{1}{2}}.$$



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$$\mathcal{L}_1^L(\Phi(u)) \in X \quad ?$$



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Yes, if $r \geq RD_-(n, \rho)$. Then, fixed-point argument gives local existence!



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Proposition

Let $u \in L^r_{\frac{n}{2r}-\frac{1}{\rho}}$ mild solution to (RD). Then

$$u \in Z^{\tilde{r}, \infty}_{\frac{n}{2\tilde{r}}-\frac{1}{\rho}}, \quad \tilde{r} \in (RD_-(n, \rho), RD_+(n, \rho)), \quad \tilde{r} \geq p.$$



Existence of weak solutions to (RD)

First consequence:

Proposition

There exists $u \in Z_{\frac{n}{2r}-\frac{1}{p}}^{r,\infty}$ weak solution to (RD) satisfying $\nabla u \in Z_{\frac{n}{2p}-\frac{1}{p}-\frac{1}{2}}^{p,2}$.

Proof.

Freeze mild solution u and use linear theory for weak solutions.
Bootstrapping result (for mild) justifies applicability! □



“Hypercontractive” SIOs

Standard case: $p, q \in (1, \infty]$, $\beta > -1$

$$S: L^q_\beta \rightarrow L^q_{\beta+\kappa} + \text{kernel} + \text{decay} \implies S: Z^{p,q}_\beta \rightarrow Z^{p,q}_{\beta+\kappa}$$



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Example: $S = \mathcal{L}^L_1$, $\kappa = 1$, $q = 2$ (Auscher–Hou)



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Example: $S = \mathcal{L}_1^L$, $\kappa = 1$, $q = 2$ (Auscher–Hou)

Used for mild existence: $\mathcal{L}_1^L: L_{\beta}^q \rightarrow L_{\beta+1-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}^r$, $q \leq r \leq q^{**}$



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Used for mild existence: $\mathcal{L}^L_1: L^q_\beta \rightarrow L^r_{\beta+1-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}$, $q \leq r \leq q^{**}$

Theorem

Let $p, q \in (1, \infty]$, $\beta > -1$.

$$S: L^q \rightarrow L^r_{\kappa-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})} + \text{kernel} + \text{decay} \implies S: Z^{p,q}_\beta \rightarrow Z^{p,r}_{\beta+\kappa}$$



Bootstrapping regularity

Recall from previous slide:

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Let $u \in L^r_{\frac{n}{2r}-\frac{1}{\rho}}$ mild solution to (RD). Then

$$u \in Z^{\tilde{r}, \infty}_{\frac{n}{2\tilde{r}}-\frac{1}{\rho}}, \quad \tilde{r} \in (RD_-(n, \rho), RD_+(n, \rho)), \quad \tilde{r} \geq p.$$

Sketch of proof: by iteration (on the board)



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Then: at most one weak solution

Ingredients:

- linear uniqueness of weak solutions
- bootstrapping for weak solutions



I was facing a Great Wall...



Bootstrapping for weak solutions

Super-linear reverse Hölder inequality:

$$\left(\iint_B |s^{\frac{1}{\rho}} u|^{2^*} \right)^{\frac{1}{2^*}} \lesssim \iint_{2B} |s^{\frac{1}{\rho}} u| + \left(\iint_{2B} |s^{\frac{1}{\rho}} u|^{2^*(1+\rho)} \right)^{\frac{1}{2^*}} \quad (\text{RH})$$



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Theorem

For some $\theta > 1$, the inequality (RD) self-improves to

$$\left(\iint_B |s^{\frac{1}{\rho}} u|^{2^*} \right)^{\frac{1}{2^*}} \lesssim \left(\iint_{\lambda B} |s^{\frac{1}{\rho}} u|^q \right)^{\frac{\theta}{q}} + \left(\iint_{\lambda B} |s^{\frac{1}{\rho}} u|^q \right)^{\frac{1}{q}},$$

where $q \in (RD_-(n, \rho), 2_*(1 + \rho))$.



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where $q \in (RD_-(n, \rho), 2_*(1 + \rho))$.

Consequence: u self-improves in weighted Z -spaces



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- Hypercontractivity is key for non-linear problems
- Iteration arguments using PDE $\implies$ regularity properties
- Regularity of solutions $\implies$ mild-weak transference principle
- Super-linear RH inequalities can also self-improve! Yields self-improvement of weak solutions.



Thanks for your attention!

A digital version of this presentation can be found here:

