Sharp geometric conditions for Sobolev extension operators

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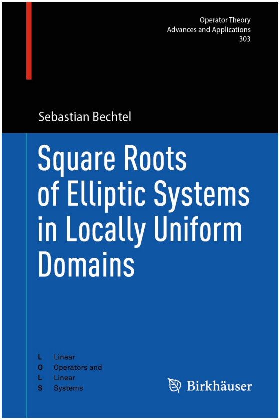

joint work (in part) with
R. Brown (Lexington), R. Haller (Darmstadt), P. Tolksdorf (KIT)

Motivation

Let $O \subseteq \mathbb{R}^d$ open.

Classical question: Does there exist $E \colon W^{1,p}(O) \to W^{1,p}(\mathbb{R}^d)$ linear & bounded with $Ef = f$ on O?

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Question

What happens if we impose a Dirichlet boundary condition?

Motivation

Define $W_0^{1,p}(O)$ as closure of $C_0^\infty(O)$-functions in $W^{1,p}(O)$.

$\implies$ $E\colon W_0^{1,p}(O) \to W^{1,p}(\mathbb{R}^d)$ linear & bounded always exists: Just extend by zero!

Motivation

Define $W^{1,p}_0(O)$ as closure of $C^\infty_0(O)$-functions in $W^{1,p}(O)$.

$\Longrightarrow$ $E\colon W^{1,p}_0(O) \to W^{1,p}(\mathbb{R}^d)$ linear & bounded always exists: Just extend by zero!

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What happens in between natural and Dirichlet boundary conditions?

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Question

What happens in between natural and Dirichlet boundary conditions?

That is to say: test functions stay away only from some boundary part $D \subseteq \partial O$.

Which sharp geometric condition to impose in $N = \partial O \setminus D$?

Outline

Let $O \subseteq \mathbb{R}^d$ open, $D \subseteq \partial O$ closed.

1. Construction of a $W^{1,p}_D(O)$ extension operator with condition in the spirit of Jones. Joint work R.M. Brown, R. Haller, and P. Tolksdorf. Appeared in **AIF**.
2. construction of a $W^{s,p}_D(O)$ extension operator, $s \in (0,1)$, using a density condition. Appeared in **Arch. Math.**

Part 1: extension operator for $W^{1,p}_D(O)$

Review of Jones' result

Setup:

- Whitney decomposition of O and $\mathbb{R}^d \setminus \overline{O}$
  $\rightsquigarrow$ interior cubes W_i and exterior cubes W_e

For simplicity: assume O unbounded and connected

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Then define E via

$$Ef = \sum_{Q \in W_e} (f)_{Q^*} \varphi_Q \quad \text{on } \mathbb{R}^d \setminus \overline{O}.$$

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Implies

$$\|\nabla Ef\|_{p,R} \leq \sum_{\substack{Q \in W_e \\ Q \cap R \neq \emptyset}} \|(f)_{Q^*} - (f)_{R^*}\|_{p,R} \quad \underbrace{\ell(Q)^{-1}}_{\text{need to compensate}}.$$

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Idea

Use Poincaré type estimate for $\|(f)_{Q^*} - (f)_{R^*}\|_{p,R}$.

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Definition

Call O an (ε, δ)-domain, if all $x, y \in O$ with $|x - y| < \delta$ can be connected by path γ in O satisfying

$$(a)\ len(\gamma) \leq \varepsilon^{-1}|x - y| \quad (b)\ d(z, \partial O) \geq \frac{\varepsilon|x - z||y - z|}{|x - y|} \quad z \in \gamma.$$

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Poincaré over this chain implies

$$\|(f)_{Q^*} - (f)_{R^*}\|_{p,R} \lesssim \ell(Q)\|\nabla f\|_{p,\text{chain}} \checkmark$$

Towards mixed boundary conditions

Assumption

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Use Whitney decomposition of $\mathbb{R}^d \setminus N$ as interior cubes W_i?

- metric properties of interior and exterior cubes become incompatible!
- path condition gives no information on interior cubes outside O...

New definition of exterior cubes

Put

$$W_{e,\mathrm{new}} = \{Q \in W_e : d(Q,N) < Bd(Q,D)\}$$

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- B large $\rightsquigarrow$ angle between sector and D small

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- B large $\rightsquigarrow$ angle between sector and D small
- upshot: use Dirichlet Poincaré instead ☺

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So what if path from Assumption runs out of O?

- Introduce "quasi-hyperbolic distance condition".
- Consequence: Can go back to O in an "efficient" way.
- Can always construct interior cubes intersecting O this way ☺

Part 2: extension operator for $W^{s,p}_D(O)$, where $s \in (0,1)$

Fractional Sobolev spaces – pure Neumann

Let $s \in (0,1)$. The space $W^{s,p}(O)$ consists of f measurable with

$$\|f\|_{s,p}^p = \|f\|_p^p + \int_{\substack{x,y \in O \\ |x-y|<1}} \left| \frac{f(x) - f(y)}{|x-y|^s} \right|^p \frac{dx\,dy}{|x-y|^d} < \infty.$$

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Zhou's result

There exists linear extension operator $\iff$ O satisfies interior thickness condition

Here, call O *interior thick*, if

$$\exists C>0\ \forall x\in O\ \forall r\in(0,1]: \quad |B(x,r)\cap O| \geq C|B(x,r)|.$$

Fractional Sobolev spaces – mixed BC

Define subspace $W^{s,p}_D(O)$ of $W^{s,p}(O)$ using condition

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Construct linear extension operator $W_D^{s,p}(O) \to W^{s,p}(\mathbb{R})$ using *only* geometric quality in N.

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$\rightsquigarrow$ assume *thickness condition in* N as follows:

$$\exists C > 0 \ \forall x \in N \ \forall r \in (0,1] : \quad |B(x,r) \cap O| \geq C|B(x,r)|.$$

Strategy for our construction

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- Construct suitable $\mathbf{O} \supseteq O$ interior thick
  $\rightsquigarrow$ use thickness in N
- Extend from O to $\mathbf{O}$ by zero
  $\rightsquigarrow$ use fractional Hardy term
- Use Zhou's result on $\mathbf{O}$.

Construction of $\mathbf{O}$

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Define $\mathbf{O} = O \cup \Big(\bigcup_{Q \in \Sigma} Q \setminus D \Big)$. Claim: $\mathbf{O}$ is interior thick.

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- r small compared to size of Q ✓
- r large compared to size of Q: Whitney $\implies$ ball intersects N ✓

Extension by zero

Let $f \in W^{s,p}_D(O)$ and F its zero extension to $\mathbf{O}$.

Need to estimate

$$\|F\|^p_{s,p} = \|f\|^p_{s,p} + 2 \int_{\substack{x \in O, y \in (\mathbf{O} \setminus O) \\ |x-y|<1}} \left| \frac{f(x)}{|x-y|^s} \right|^p \frac{dx\, dy}{|x-y|^d} + 0.$$

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Claim: One has $|x-y| \geq \frac{1}{2} d(x, D)$. Then:

$$\int_{\substack{x \in O, y \in \mathbf{O} \\ |x-y|<1}} \left| \frac{f(x)}{|x-y|^s} \right|^p \frac{dx\, dy}{|x-y|^d} \lesssim \int_{x \in O} \left| \frac{f(x)}{d(x,D)^s} \right|^p dx \checkmark$$

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- Fix $z \in \partial O$ on line segment from x to y.
- $z \in N$ implies

$$d(Q, N) \leq |y - z| \leq |x - y| < diam(Q) \leq d(Q, N). \lightning$$

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$$d(Q, N) \leq |y - z| \leq |x - y| < diam(Q) \leq d(Q, N). ⚡$$

- Hence $z \in D$ and $|x - y| \geq |x - z| \geq d(x, D)$. ✓

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- Can pick $z \in Q \cap D$.
- Estimate

$$|x - z| \leq |x - y| + |y - z| \leq |x - y| + diam(Q) \leq 2|x - y|.$$

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- Estimate

$$|x - z| \leq |x - y| + |y - z| \leq |x - y| + diam(Q) \leq 2|x - y|.$$

- Conclude $d(x, D) \leq |x - z| \leq 2|x - y|$.

Thanks for your attention!

A digital version of this presentation can be found here: