The Poincaré inequality in uniform domains

Sebastian Bechtel
TU Delft

based on joint work with
Brown – Haller-Dintelmann – Tolksdorf

FSA - 20~~20~~22

Motivation: Riesz transforms on rough domains

Consider: $L = -\mathrm{div}(A\nabla + b) + c\nabla + d$

Let $p > 2$.

Question: $\nabla L^{-1/2} : L^p \to L^p$ bounded ?

Motivation: Riesz transforms on rough domains

Consider: $L = -\mathrm{div}(A\nabla + b) + \cancel{c\nabla + d}$

Let $p>2$.

Question: $\nabla L^{-1/2} : L^p \to L^p$ bounded ?

Characterization on $\mathbb{R}^d$: (Auscher '07, Blunck-Kunstmann '04, Hofmann-Martell '03)

$\nabla L^{-1/2}$ L^p-bounded $\Longleftrightarrow$ $\{t \nabla e^{-t^2 L}\}_{t>0}$ satisfies $L^2 \to L^p$ ODEs

Motivation: Riesz transforms on rough domains

Consider: $L = -\mathrm{div}(A\nabla + b) + \cancel{c\nabla + d}$

Let $p > 2$.

Question: $\nabla L^{-1/2} : L^p \to L^p$ bounded ?

Characterization on $\mathbb{R}^d$: (Auscher '07, Blunck-Kunstmann '04, Hofmann-Martell '03)

$$\nabla L^{-1/2} \ L^p\text{-bounded} \iff \{t \nabla e^{-t^2 L}\}_{t>0} \text{ satisfies } L^2 \to L^p \text{ ODEs}$$

What about domains in $\mathbb{R}^d$?

Motivation: Riesz transforms on rough domains

Proof is based on extrapolation via good-λ estimates

Ingredients:

- Kato's square root problem
- Usual inequalities: Gagliardo–Nirenberg, Sobolev embeddings, ...
- Conservation property: "$e^{-tL} 1 = 1$"
- local Poincaré inequality:

$B \subseteq \mathbb{R}^d$ ball of radius $r(B)$

$$\left(\fint_B |f - f_B|^p \right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \right)^{1/p}$$

Motivation: Riesz transforms on rough domains

Proof is based on extrapolation via good-λ estimates

Ingredients:

- Kato's square root problem
- Usual inequalities: Gagliardo–Nirenberg, Sobolev embeddings, ...
- Conservation property: "$e^{-tL} 1 = 1$"
- local Poincaré inequality:

  $B \subseteq \mathbb{R}^d$ ball of radius $r(B)$

$$\left(\fint_B |f - f_B|^p \right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \right)^{1/p}$$

consequence of non-commutativity of ∇ and L

Motivation: Riesz transforms on rough domains

Proof is based on extrapolation via good-λ estimates

Ingredients:

- Kato's square root problem
- Usual inequalities: Gagliardo–Nirenberg, Sobolev embeddings, ...
- Conservation property: "$e^{-tL} 1 = 1$" $\Rightarrow$ pure Neumann BC

  then: OK
- local Poincaré inequality:

  $B \subseteq \mathbb{R}^d$ ball of radius $r(B)$

$$\left(\fint_B |f - f_B|^p \right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \right)^{1/p}$$

Motivation: Riesz transforms on rough domains

Proof is based on extrapolation via good-λ estimates

Ingredients:

- Kato's square root problem → with Neumann BC: in (ε,δ)-domains
  (B. — Egert — Haller-Dintelmann)
- Usual inequalities: Gagliardo–Nirenberg, Sobolev embeddings, …
- Conservation property: "$e^{-tL}1 = 1$"
- local Poincaré inequality:

$B \subseteq \mathbb{R}^d$ ball of radius $r(B)$

$$\left(\fint_B |f - f_B|^p \right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \right)^{1/p}$$

Motivation: Riesz transforms on rough domains

Proof is based on extrapolation via good-λ estimates

Ingredients:

- ∗ Kato's square root problem
- ∗ Usual inequalities: Gagliardo–Nirenberg, Sobolev embeddings, ... OK
- ∗ Conservation property: "$e^{-tL} 1 = 1$"
- ∗ local Poincaré inequality:

$B \subseteq \mathbb{R}^d$ ball of radius $r(B)$

$$\left(\fint_B |f - f_B|^p \right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \right)^{1/p}$$

Motivation: Riesz transforms on rough domains

Proof is based on extrapolation via good-λ estimates

Ingredients:

* Kato's square root problem

* Usual inequalities: Gagliardo–Nirenberg, Sobolev embeddings, ...

* Conservation property: "$e^{-tL} 1 = 1$"

* local Poincaré inequality:

$B \subseteq \mathbb{R}^d$ ball of radius $r(B)$

$$\left(\fint_B |f - f_B|^p \right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \right)^{1/p}$$

?

Poincaré inequality: approach via extension operator

Let:

* Λ extension operator for $O \subseteq \mathbb{R}^d$,
* B ball in $\mathbb{R}^1$ centered in $\overline{O}$.

Calculate:

$$\left(\fint_{B\cap O} |f - f_{B\cap O}|^p \right)^{1/p} \overset{\text{thickness of } O}{\lesssim} \left(\fint_{B} |\Lambda f - (\Lambda f)_B|^p \right)^{1/p} + \left(\fint_{B} |(\Lambda f)_B - f_{B\cap O}|^p \right)^{1/p}$$

Poincaré inequality: approach via extension operator

Let:

$*$ Λ extension operator for $O \subseteq \mathbb{R}^d$,

$*$ B ball in $\mathbb{R}^1$ centered in $\overline{O}$.

Calculate:

$$\Big(\fint_{B\cap O} |f - f_{B\cap O}|^p\Big)^{1/p} \overset{\text{thickness of } O}{\lesssim} \underbrace{\Big(\fint_{B} |\Lambda f - (\Lambda f)_B|^p\Big)^{1/p}} + \Big(\fint_{B} |(\Lambda f)_B - f_{B\cap O}|^p\Big)^{1/p}$$

$$\overset{\text{Poincaré}}{\lesssim} r(B) \Big(\fint_{B} |\nabla(\Lambda f)|^p\Big)^{1/p}$$

$$\overset{!}{\lesssim} r(B) \Big(\fint_{cB\cap O} |\nabla f|^p\Big)^{1/p}$$

"local" & homogeneous estimates for Λ

Poincaré inequality: approach via extension operator

Let:

- Λ extension operator for $O \subseteq \mathbb{R}^d$,
- B ball in $\mathbb{R}^1$ centered in $\overline{O}$.

Calculate:

thickness of O

$$\left(\fint_{B\cap O} |f - f_{B\cap O}|^p\right)^{1/p} \lesssim \underbrace{\left(\fint_B |\Lambda f - (\Lambda f)_B|^p\right)^{1/p}} + \underbrace{\left(\fint_B |(\Lambda f)_B - f_{B\cap O}|^p\right)^{1/p}}$$

$$\overset{\text{Poincaré}}{\lesssim} r(B)\left(\fint_B |\nabla(\Lambda f)|^p\right)^{1/p}$$

$$\overset{!}{\lesssim} r(B)\left(\fint_{cB\cap O} |\nabla f|^p\right)^{1/p}$$

$$\leq \left(\fint_B |\Lambda f - (\Lambda f)_{B\cap O}|^p\right)^{1/p}$$

$$\overset{\text{Poincaré}}{\lesssim} r(B)\,\frac{|B|}{|B\cap O|}\left(\fint_B |\nabla(\Lambda f|^p\right)^{1/p}$$

$$\overset{!}{\lesssim} r(B)\left(\fint_{cB\cap O} |\nabla f|^p\right)^{1/p}$$

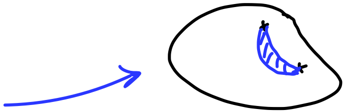
"local" & homogeneous estimates for Λ

thickness of O

Results of Jones

Recall: Call $O \subseteq \mathbb{R}^d$ an **(ε, δ)-domain**, if for $x, y \in O$ with $|x-y| \leq \delta$ there is connecting path γ with

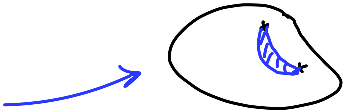
- $\ell(\gamma) \leq |x-y|/\varepsilon$,
- $d(z, \partial O) \geq \dfrac{\varepsilon |x-z||y-z|}{|x-y|}$.



Results of Jones

Recall: Call $O \subseteq \mathbb{R}^d$ an **(ε, δ)-domain**, if for $x, y \in O$ with $|x-y| \leq \delta$ there is connecting path γ with

* $\ell(\gamma) \leq |x-y|/\varepsilon$,
* $d(z, \partial O) \geq \dfrac{\varepsilon |x-z||y-z|}{|x-y|}$.



Thm: (Jones '81, Acta Math)

Let $1 \leq p < \infty$ and $O \subseteq \mathbb{R}^d$.

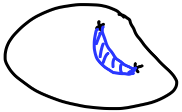
(1) O is (ε, δ) domain $\Longrightarrow$ exists $W^{\ell,p}(O)$-extension operator for each $\ell \geq 1$

(2) O is (ε, ∞) domain $\Longrightarrow$ exists $\dot{W}^{1,p}(O)$-extension operator

Results of Jones

Recall: Call $O \subseteq \mathbb{R}^d$ an **(ε, δ)-domain**, if for $x, y \in O$ with $|x-y| \leq \delta$ there is connecting path γ with

* $\ell(\gamma) \leq |x-y|/\varepsilon$,
* $d(z, \partial O) \geq \dfrac{\varepsilon |x-z||y-z|}{|x-y|}$.



Thm: (Jones '81, Acta Math)

Let $1 \leq p < \infty$ and $O \subseteq \mathbb{R}^d$.

(1) O is (ε, δ) domain $\Rightarrow$ exists $W^{k,p}(O)$-extension operator, $k \geq 1$

(2) O is (ε, ∞) domain $\Rightarrow$ exists $\dot{W}^{1,p}(O)$- extension operator

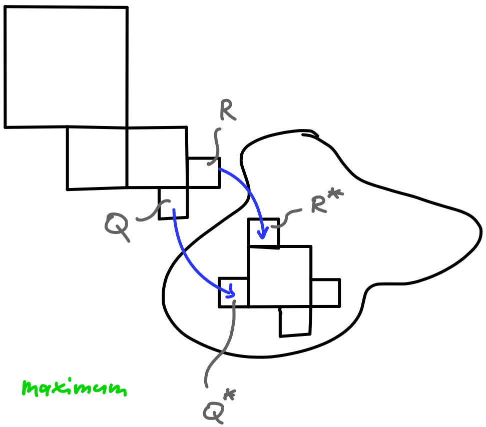
Note: operators in (1) and (2) are different!

Results of Jones

general idea:

- Whitney decompositions of O and $\mathbb{R}^d \setminus \bar{O}$ → interior and exterior cubes
- associate interior cubes to "small" exterior cubes: $Q \to Q^*$

in Jones: some maximum size ⇝ close to O

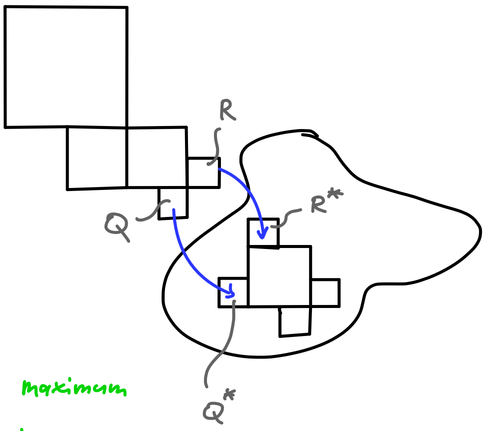


Results of Jones

general idea:

- Whitney decompositions of O and $\mathbb{R}^d \setminus \bar{O}$ → interior and exterior cubes
- associate interior cubes to "small" exterior cubes: $Q \longrightarrow Q^*$

in Jones: some maximum size ⇝ close to O

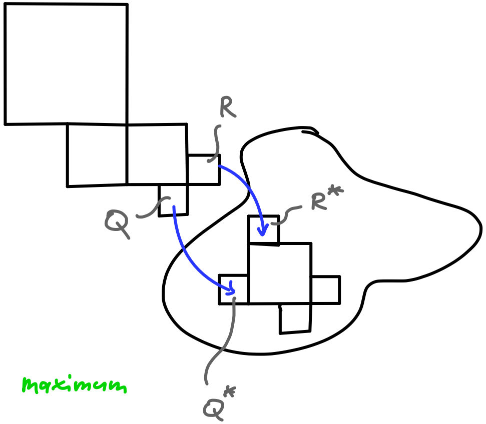


also:

- φ_Q partition of unity on exterior cubes
- $P_{Q^*}(f)$ polynomial approximation of f on Q^*

Results of Jones

general idea:



* Whitney decompositions of O and $\mathbb{R}^d \setminus \bar{O}$ → interior and exterior cubes

* associate interior cubes to "small" exterior cubes: $Q \rightarrow Q^*$

in Jones: some maximum size ⇝ close to O

also:

* φ_Q partition of unity on exterior cubes

* $P_{Q^*}(f)$ polynomial approximation of f on Q^*

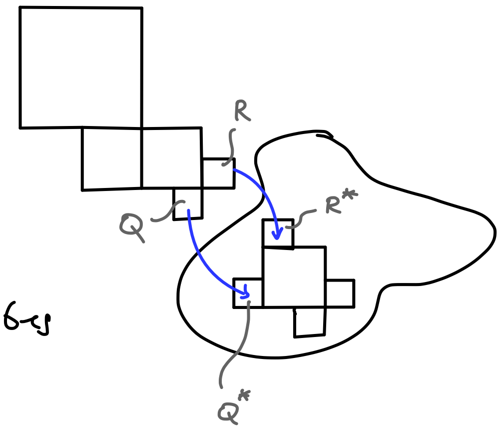
Put

$$\Lambda f = \sum_{Q \text{ small}} \varphi_Q P_{Q^*}(f) \qquad \text{on } \mathbb{R}^d \setminus \overline{O}.$$

Results of Jones

Recall:

$$\Lambda f = \sum_{Q \text{ small}} \varphi_Q P_{Q^*}(f) \qquad \text{on} \quad \mathbb{R}^d \setminus \overline{\Omega}.$$



Polynomial approximation: Sobolev & Poincaré-type estimates

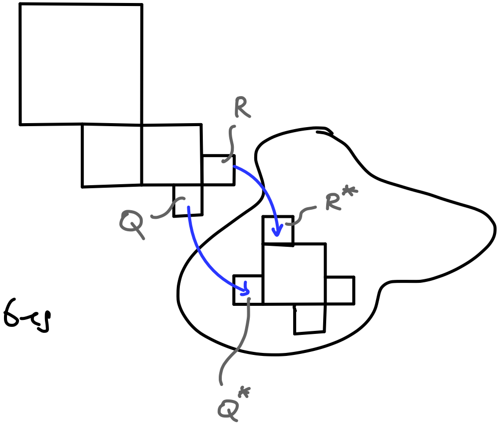
geometry: Q touches R $\Longrightarrow$ connecting chain of cubes from Q^* to R^*

$W^{k,p}$-bound on exterior cube: estimate (at most) over connecting chain of cubes

Results of Jones

Recall:

$$\Lambda f = \sum_{Q \text{ small}} \varphi_Q P_{Q^*}(f) \qquad \text{on} \quad \mathbb{R}^d \setminus \overline{\Omega}.$$



Polynomial approximation: Sobolev & Poincaré-type estimates

geometry: Q touches R ⟹ connecting chain of cubes from Q^* to R^*

$W^{k,p}$-bound on exterior cube: estimate (at most) over connecting chain of cubes

1st Observation: chain has fixed length & cubes have comparable size

⟹ chain contained in enlarged cube

⟹ Λ local

Results of Jones

Sizes:

	inhomogeneous
rad(O) "minimal size"	$\geqslant 1$
diam(O) "maximal size"	arbitrary
δ	$\leqslant 1$

exterior cubes: finite maximal size depending on ε and δ

Results of Jones

Sizes:

		inhomogeneous	homogeneous
	rad(O) "minimal size"	≥ 1	∞
			$\Uparrow$ geometry
	diam(O) "maximal size"	arbitrary	wlog ∞
			$\Uparrow$ transformation
	δ	≤ 1	∞

exterior cubes:

finite maximal size depending on ε and δ	all cubes

Results of Jones

Sizes:

	inhomogeneous	homogeneous
$\mathrm{rad}(O)$ "minimal size"	≥ 1	∞
		$\Uparrow$ geometry
$\mathrm{diam}(O)$ "maximal size"	arbitrary	wlog ∞
		$\Uparrow$ transformation
δ	≤ 1	∞

exterior cubes:

inhomogeneous	homogeneous		
finite maximal size depending on ε and δ	all cubes	$\}$ $\Rightarrow$	constructions are different

Improvements by Chua:

(* weighted spaces)

* different choice of polynomials $\Longrightarrow$ homogeneous estimates of order $k \geqslant 2$

(also estimates of "lower order")

Improvements by Chua:

(* weighted spaces)

* different choice of polynomials $\Longrightarrow$ homogeneous estimates of order $k \geq 2$
(also estimates of "lower order")

2nd observation: Fix $k \geq 1$. $P_{Q^*}(f)$ well-defined for $f \in L^1_{loc}$

$\Longrightarrow \Lambda$ defined on L^1_{loc} & semi-universal:

$\Longleftrightarrow W^{l,p}$- bounded
for $0 \leq l \leq k$

Improvements by Chua:

(* weighted spaces)

* different choice of polynomials $\Longrightarrow$ homogeneous estimates of order $k \geqslant 2$
(also estimates of "lower order")

2nd observation: Fix $\ell \geqslant 1$. $P_{Q^*}(f)$ well-defined for $f \in L^1_{loc}$

$\Longrightarrow$ Λ defined on L^1_{loc} & semi-universal:

$\Longleftrightarrow$ $W^{\ell,p}$-bounded for $0 \leq \ell \leq k$

* choice of cubes depends on δ and rad(O)

Scaling & homogeneous estimates

Introduce coupling between $\mathrm{rad}(O)$ and δ:

$$\mathrm{rad}(O) \geqslant \lambda \delta \qquad (\text{for some } 0 < \lambda < \infty).$$

Construct exterior cubes with condition

$$\ell(Q) \leq A\delta \qquad (\text{for } A>0 \text{ sufficiently small}).$$

Scaling & homogeneous estimates

Introduce coupling between rad(O) and δ:

$$\mathrm{rad}(O) \geqslant \lambda \delta \qquad (\text{for some } 0 < \lambda < \infty).$$

Construct exterior cubes with condition

$$\ell(Q) \leq A\delta \qquad (\text{for } A > 0 \text{ sufficiently small}).$$

Thm.: (B. – Brown – Haller-Dintelmann – Tolksdorf)

Let $k \geqslant 1$.

Then Λ is

- semi-universal $W^{k,p}$-extension operator for $1 \leq p < \infty$,
- admits local estimats,
- 3rd observation admits homogeneous estimates in δ-tube around O.

Scaling & homogeneous estimates

Recall the coupling $\mathrm{rad}(O) \geqslant \lambda\delta$ between $\mathrm{rad}(O)$ and δ.

Cor.: (B. – Brown – Haller-Dintelmann – Tolksdorf)

Let $1 \leq p < \infty$, $k \geqslant 1$ and $O \subseteq \mathbb{R}^d$ an unbounded (ε, ∞)-domain

Then Λ is $\dot{W}^{k,p}$-bounded and local.

Scaling & homogeneous estimates

Recall the coupling $\mathrm{rad}(O) \geqslant \lambda \delta$ between $\mathrm{rad}(O)$ and δ.

Cor.: (B. – Brown – Haller-Dintelmann – Tolksdorf)

Let $1 \leq p < \infty$, $k \geqslant 1$ and $O \subseteq \mathbb{R}^d$ an unbounded (ε, ∞)-domain

Then Λ is $\dot{W}^{k,p}$-bounded and local.

Cor.:

Let $1 \leq p < \infty$ and $O \subseteq \mathbb{R}^d$ an unbounded (ε, ∞)-domain. Then O admits local p-Poincaré inequalities.

Thank you

for your attention!